Analysis and design of composite structures

Class notes

2. Lamina: macromechanical behavior

Fundamentals of continuum mechanics

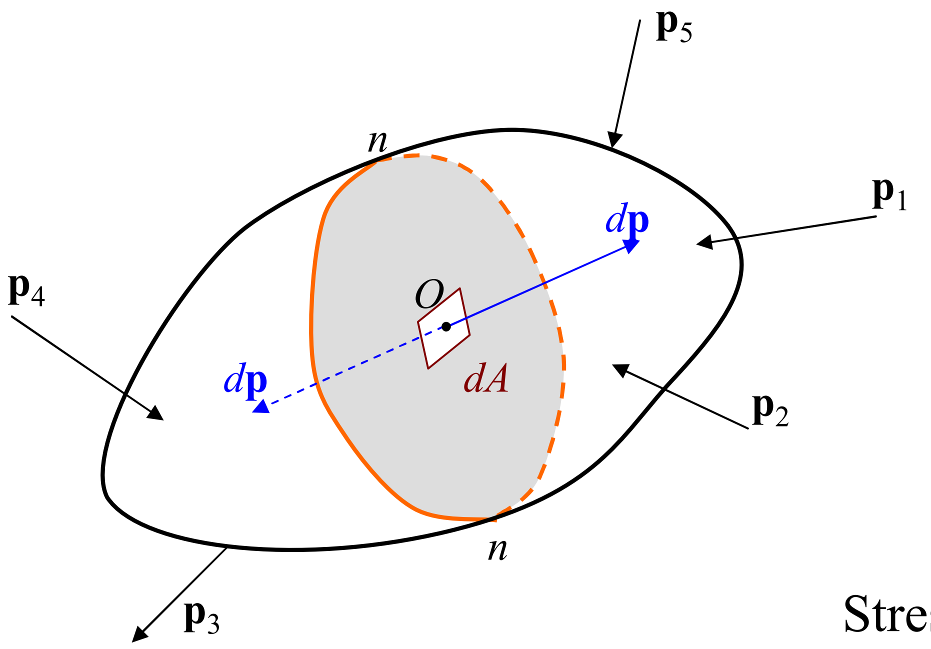
Ignore infinitesimal molecular details

Any solid is formed by material points continuously distributed

Mathematical manipulations are considerable

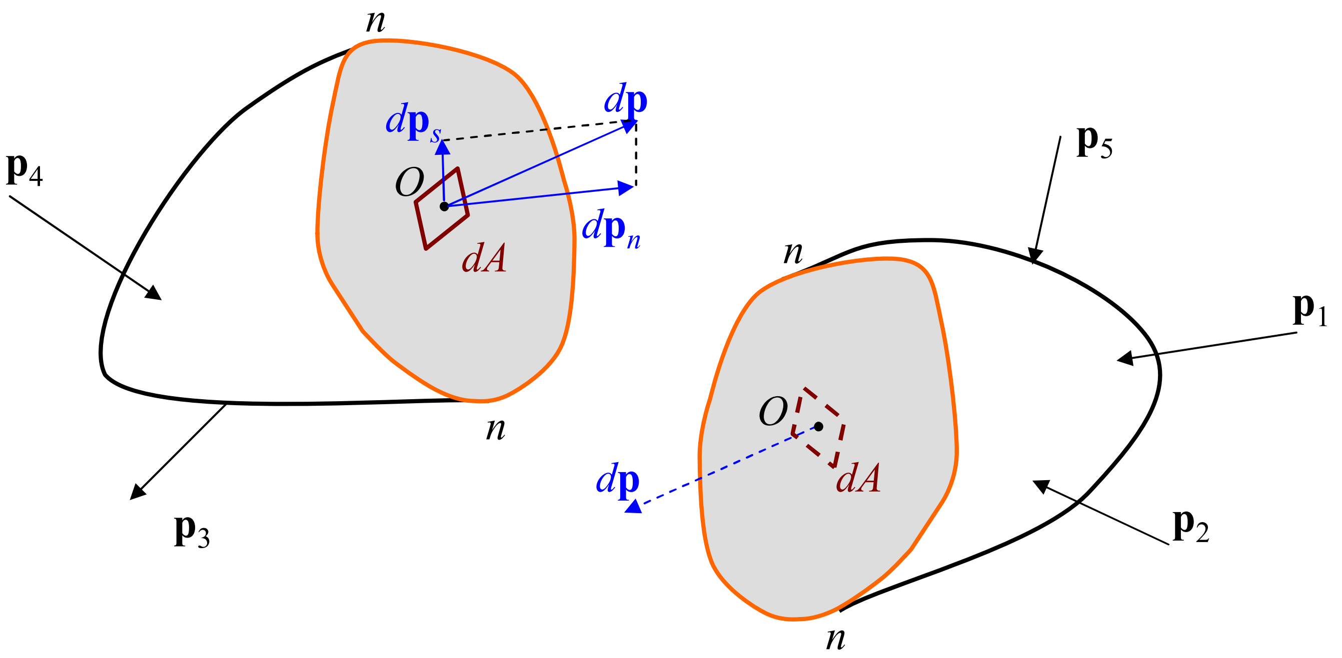
2.1. Stress transformations

Internal force in an arbitrary point of a 3D body



Stress vector: $$\mathbf{t} = \lim_{dA \to 0} \frac{d\mathbf{p}}{dA}$$

Internal force components

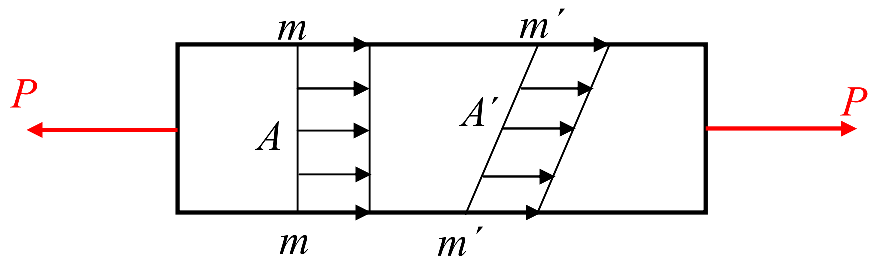


$$\sigma = \lim_{dA \to 0} \frac{dP_n}{dA}$$

$$\tau = \lim_{dA \to 0} \frac{dP_s}{dA}$$

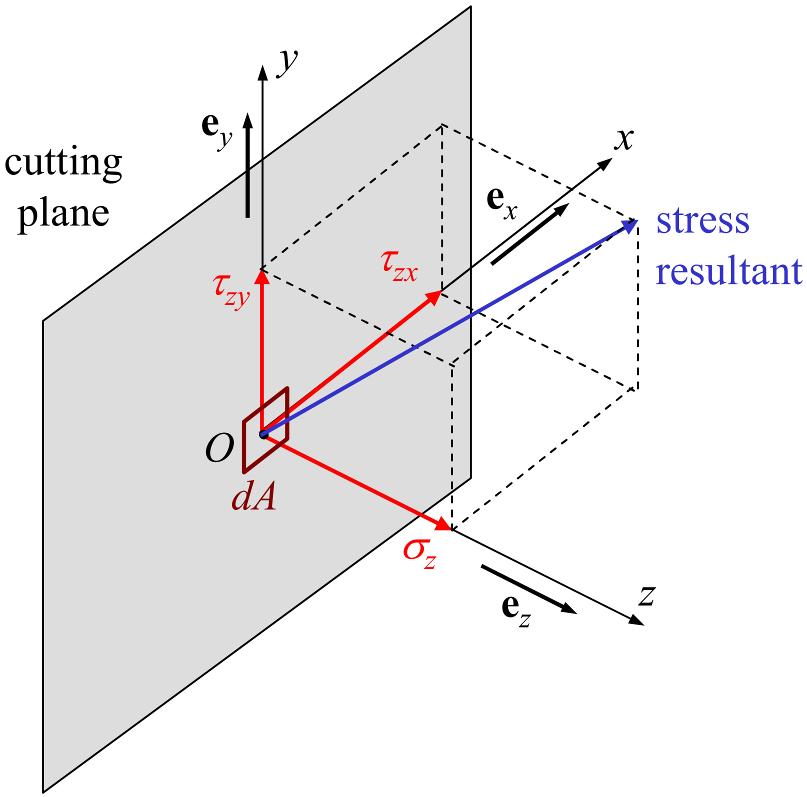
stress resultant = $\sqrt{\sigma^2 + \tau^2}$

Different stresses on different cutting planes



The stress state of a point is unique. However, it may be described in different reference systems.

Different stresses on different cutting planes

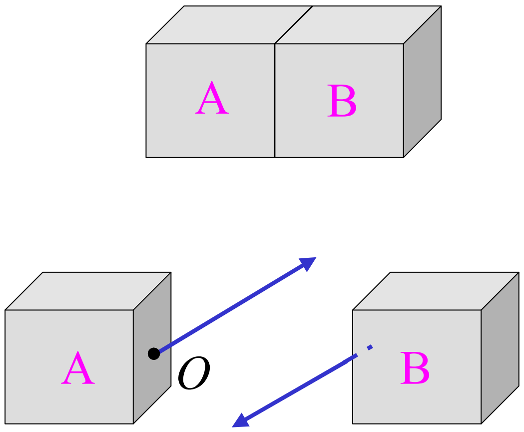


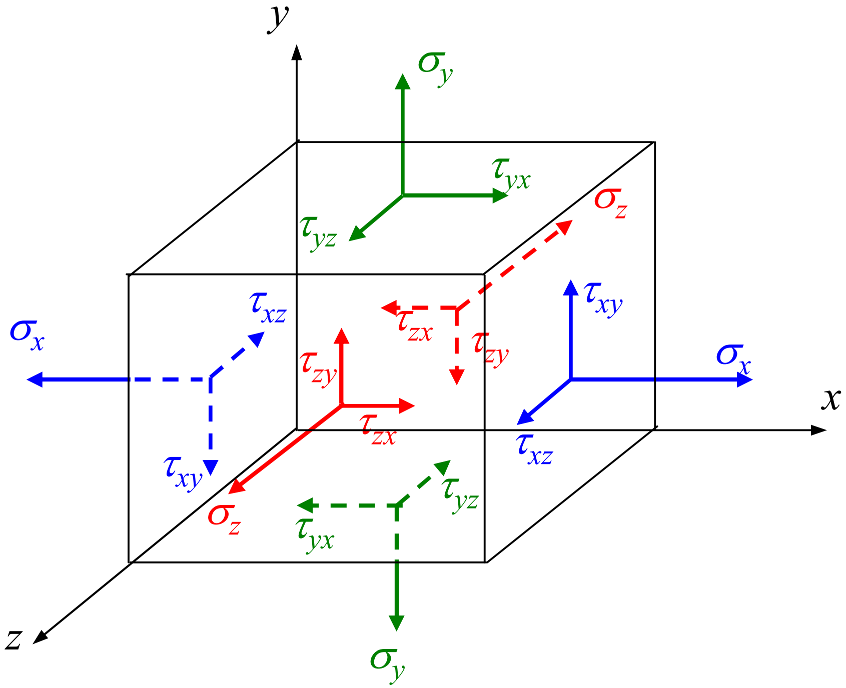
Shear stress on plane z: τ_{zx} and τ_{zy}

Normal stress on plane z: σ_z

$$\mathbf{t} = \tau_{zx}\mathbf{e}_x + \tau_{zy}\mathbf{e}_y + \sigma_z\mathbf{e}_z$$

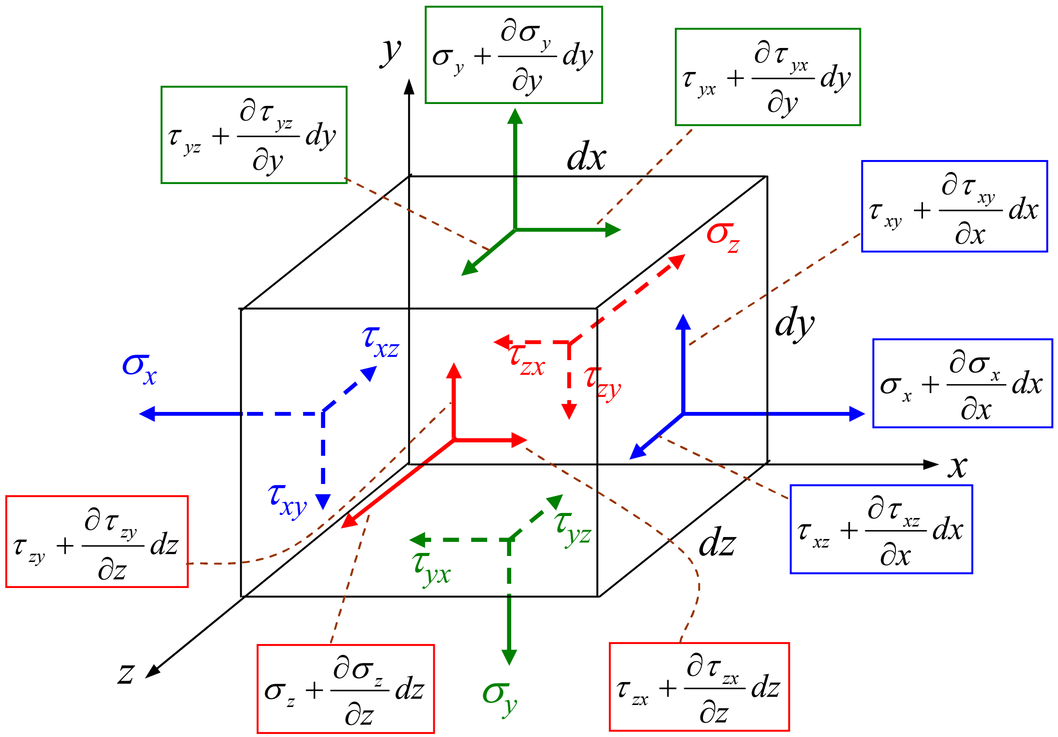
Sign convention and notation for stress state in a point





$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Equilibrium equations (infinitesimal volume)



Moment equilibrium about *z*

$$\left(\tau_{xy}+\frac{\partial \tau_{xy}}{\partial x}dx\right)(dydz)dx-\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)(dxdz)dy+\left(\tau_{zy}+\frac{\partial \tau_{zy}}{\partial z}dz\right)(dxdy)\frac{dx}{2}-\tau_{zy}(dxdy)\frac{dx}{2}+$$

$$\tau_{zx}(dxdy)\frac{dy}{2}-\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}dz\right)(dxdy)\frac{dy}{2}+$$

$$\left(\sigma_{y}+\frac{\partial \sigma_{y}}{\partial y}dy\right)(dxdz)\frac{dx}{2}-\sigma_{y}(dxdz)\frac{dx}{2}+\sigma_{x}(dydz)\frac{dy}{2}-\left(\sigma_{x}+\frac{\partial \sigma_{x}}{\partial x}dx\right)(dydz)\frac{dy}{2}=0$$

$$\Downarrow$$

$$\tau_{xy}=\tau_{yx}$$

Similarly, moment equilibrium about *x* and *y*

$$\tau_{yz}=\tau_{zy} \qquad \tau_{xz}=\tau_{zx}$$

Force equilibrium: *x* axis

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dydz - \sigma_x dydz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dydz - \tau_{yx} dydz +$$

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy - \tau_{zx} dxdy + b_x dxdydz = 0 \quad \Longrightarrow \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0$$

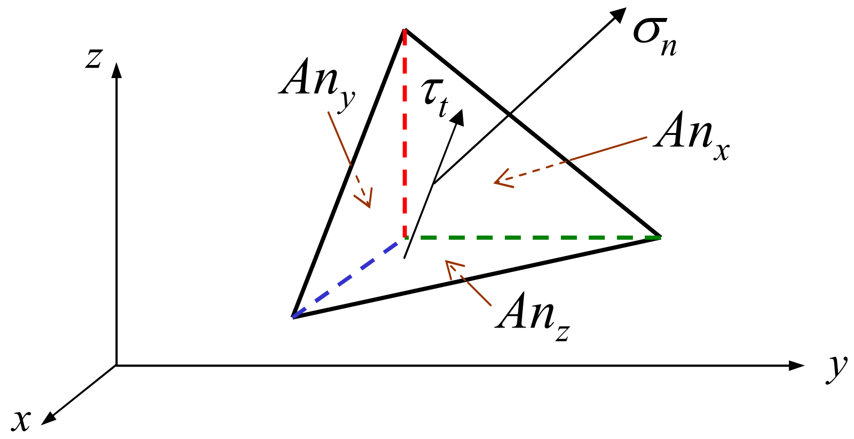
Force equilibrium: *y* axis

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0$$

Force equilibrium: *z* axis

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + b_z = 0$$

Change of reference system



σ_n direction given by normal unity vector $\{n\} = \{n_x\ n_y\ n_z\}^T$

τ_t direction given by tangent unity vector $\{t\} = \{t_x\ t_y\ t_z\}^T$

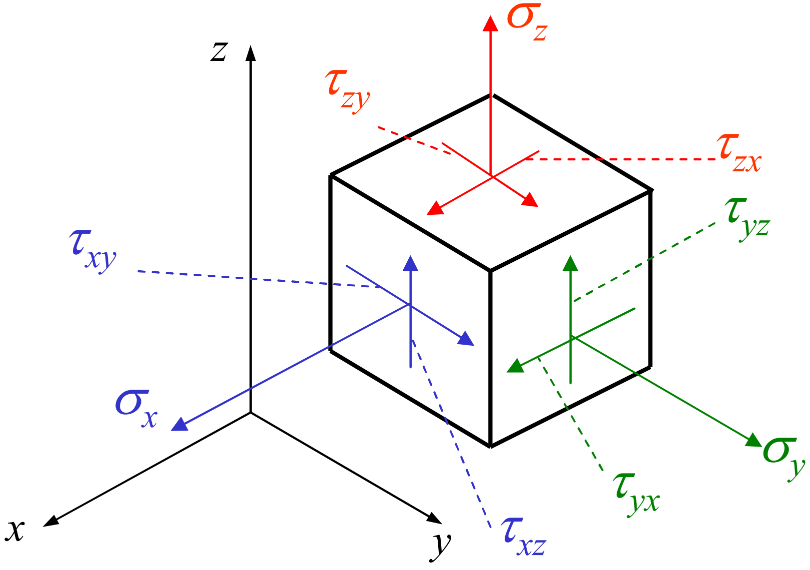
$$
\begin{aligned}
\sigma_n A = {} & (\sigma_x A n_x) n_x + (\tau_{xy} A n_x) n_y + (\tau_{xz} A n_x) n_z \\
& (\sigma_y A n_y) n_y + (\tau_{yx} A n_y) n_x + (\tau_{yz} A n_y) n_z \\
& (\sigma_z A n_z) n_z + (\tau_{zx} A n_z) n_x + (\tau_{zy} A n_z) n_y
\end{aligned}
$$

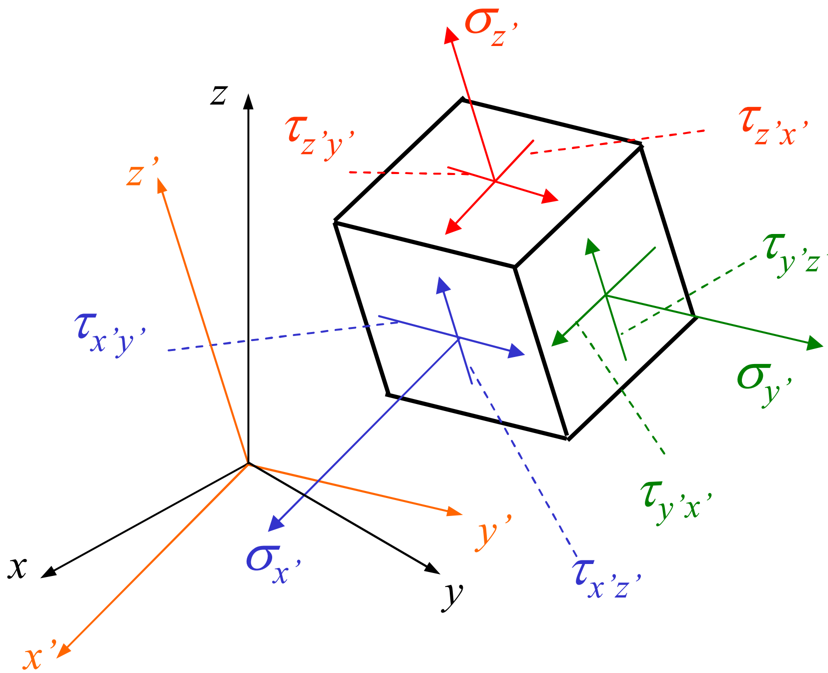
$$\sigma_n = \{n\}^T [\sigma] \{n\}$$

$$
\begin{aligned}
\tau_t = {} & \sigma_x n_x t_x + \sigma_y n_y t_y + \sigma_z n_z t_z + \\
& \tau_{xy}(n_x t_y + n_y t_x) + \\
& \tau_{xz}(n_x t_z + n_z t_x) + \\
& \tau_{yz}(n_y t_z + n_z t_y)
\end{aligned}
$$

$$\tau_t = \{t\}^T [\sigma] \{n\}$$

Change of reference system





$$[\sigma'] = [l]^T [\sigma][l] \qquad [l] = [\{n_{x'}\} \quad \{n_{y'}\} \quad \{n_{z'}\}]$$

Principal stresses

Find extremes of $\sigma_n = \{n\}^T[\sigma]\{n\}$ subject to $\{n\}^T\{n\} = 1$.

Define extended function: $\sigma_n^* = \sigma_n - \lambda(\{n\}^T\{n\} - 1)$

$$\left.\begin{aligned}
\frac{1}{2}\frac{\partial \sigma_n^*}{\partial n_x} &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z - \lambda n_x = 0 \\
\frac{1}{2}\frac{\partial \sigma_n^*}{\partial n_y} &= \tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z - \lambda n_y = 0 \\
\frac{1}{2}\frac{\partial \sigma_n^*}{\partial n_z} &= \tau_{zx} n_x + \tau_{zy} n_y + \sigma_z n_z - \lambda n_z = 0
\end{aligned}\right\} \Rightarrow \underbrace{([\sigma] - \lambda[I])\{n\} = \{0\}}_{\text{eigenvalue problem}}$$

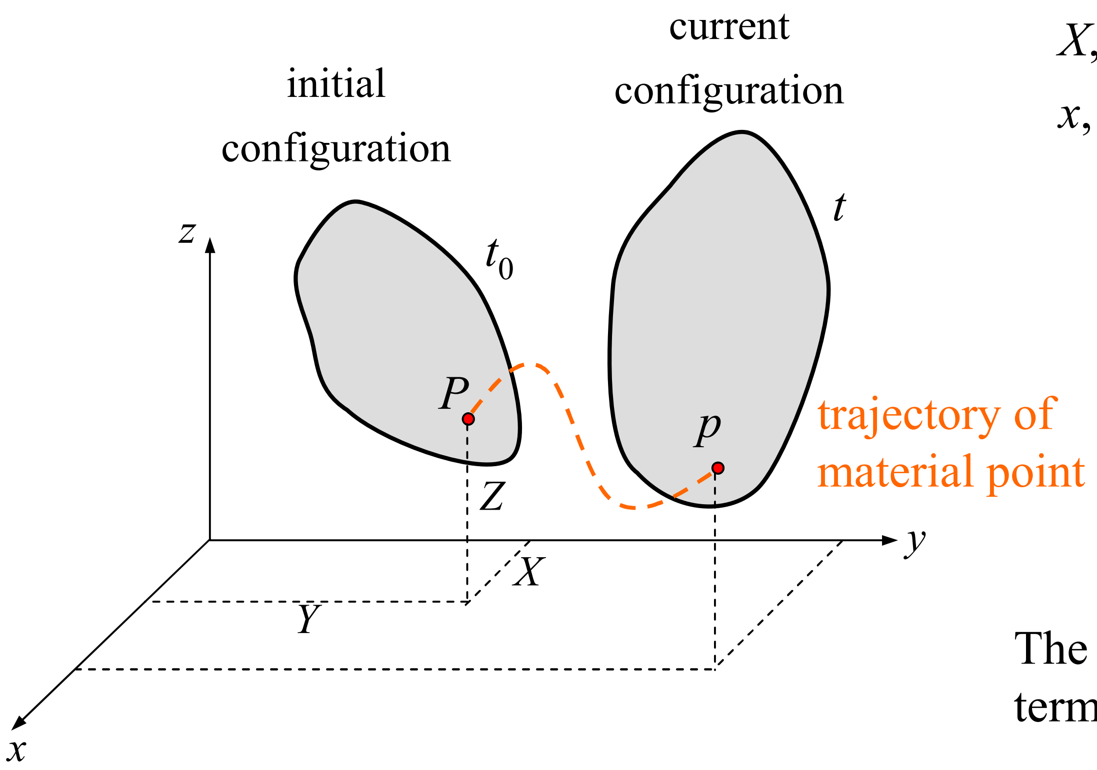
eigenvalue problem

Solution of the eigenvalue problem provides principal stresses $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$ and their respective principal directions $\{n_I\}$, $\{n_{II}\}$ and $\{n_{III}\}$.

In the principal reference system there are no shear stresses.

2.2. Strain transformations

Configuration of a body: transformation



$X, Y, Z \rightarrow$ lagrangian coordinates

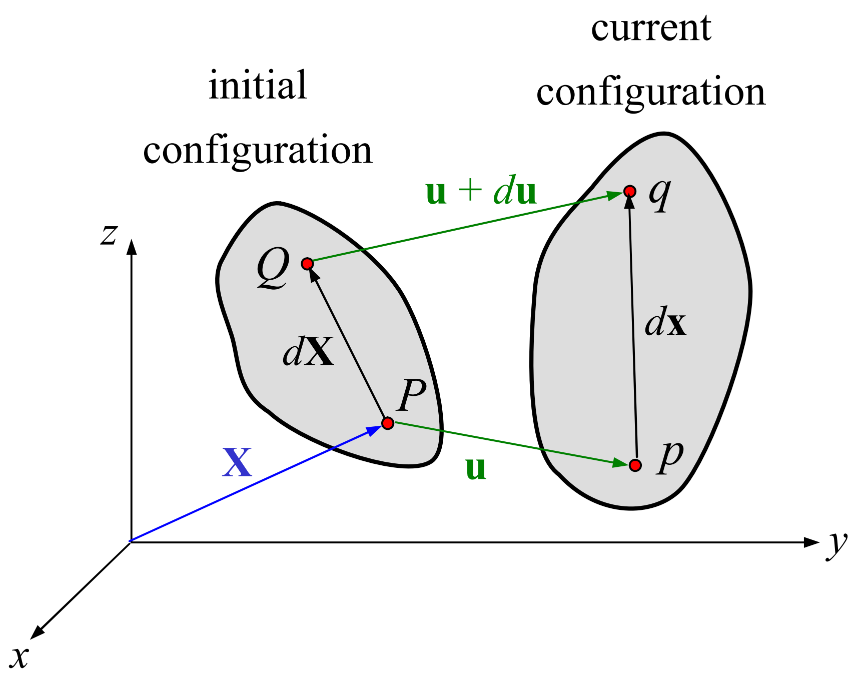
$x, y, z \rightarrow$ eulerian coordinates

$$x = x(X, Y, Z, t)$$
$$y = y(X, Y, Z, t)$$
$$z = z(X, Y, Z, t)$$

The change of configuration of a body is termed *transformation*

Transformations involving change of shape or volume are *deformations*

Transformation or deformation gradient



$$x = x(X, Y, Z, t)$$

$$y = y(X, Y, Z, t)$$

$$z = z(X, Y, Z, t)$$

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}$$

Transformation or deformation gradient

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}$$

$\mathbf{F}$ → Deformation gradient

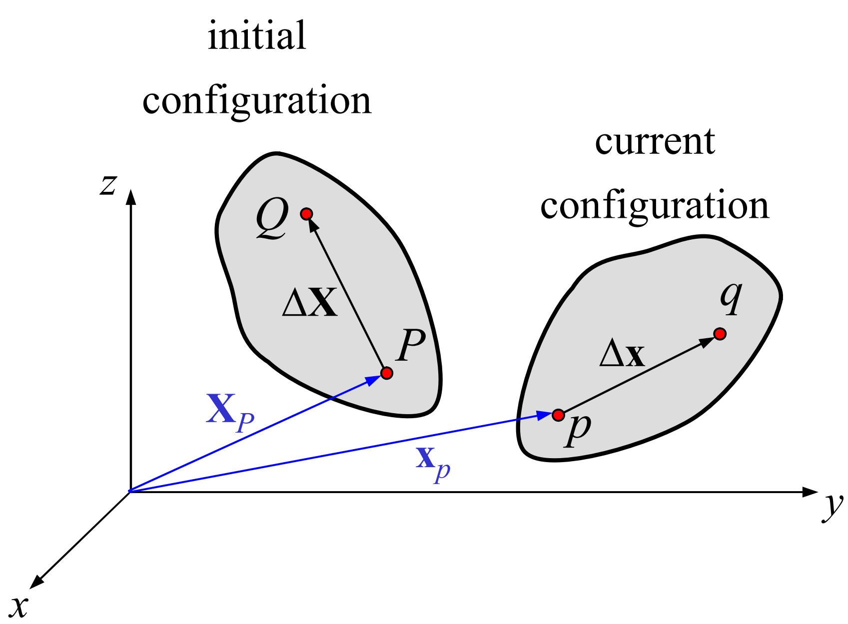
$$d\mathbf{u} = \begin{Bmatrix} du \\ dv \\ dw \end{Bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial Z} \end{bmatrix} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \mathbf{H}d\mathbf{X}$$

$$\mathbf{u} = \mathbf{x} - \mathbf{X}$$

} $\Rightarrow$

$$\mathbf{F} = \mathbf{I} + \mathbf{H}$$

Deformation gradient and rigid body motion



$$\Delta \mathbf{x} = \mathbf{F}\Delta \mathbf{X}$$

$$(\Delta \mathbf{X})^T(\Delta \mathbf{X}) = (\Delta \mathbf{x})^T(\Delta \mathbf{x}) = (\Delta \mathbf{X})^T \mathbf{F}^T\mathbf{F}(\Delta \mathbf{X})$$

$$(\Delta \mathbf{X})^T[\mathbf{I} - \mathbf{F}^T\mathbf{F}](\Delta \mathbf{X}) = 0 \Rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{I}$$

$$\mathbf{x} = \mathbf{x}_p + \mathbf{F}(\mathbf{X} - \mathbf{X}_P) = \mathbf{X}_P + \mathbf{u} + \mathbf{F}(\mathbf{X} - \mathbf{X}_P)$$

Pure translation $\mathbf{u} = \mathbf{x}_p - \mathbf{X}_P$ followed by pure rotation $\mathbf{F}(\mathbf{X} - \mathbf{X}_P)$

Green strain tensor

$$(dS)^2 = (d\mathbf{X})^T (d\mathbf{X})$$

$$(ds)^2 = (d\mathbf{x})^T (d\mathbf{x}) = (d\mathbf{X})^T \mathbf{F}^T \mathbf{F} (d\mathbf{X})$$

$$(ds)^2 - (dS)^2 = (d\mathbf{X})^T [\mathbf{F}^T \mathbf{F} - \mathbf{I}] (d\mathbf{X})$$

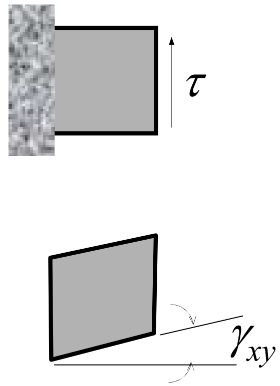
$$\frac{(ds)^2 - (dS)^2}{2(dS)^2} = \left(\frac{d\mathbf{X}}{dS}\right)^T \left(\frac{\mathbf{F}^T \mathbf{F} - \mathbf{I}}{2}\right) \frac{d\mathbf{X}}{dS} = \{n\}^T \left(\frac{\mathbf{F}^T \mathbf{F} - \mathbf{I}}{2}\right) \{n\}$$

$$[\varepsilon] = \frac{\mathbf{F}^T \mathbf{F} - \mathbf{I}}{2} = \frac{\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}}{2}$$

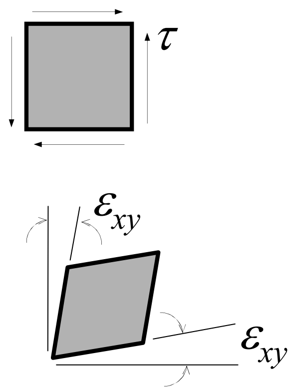
$$\{n\}^T [\varepsilon] \{n\} = \{n'\}^T [\varepsilon'] \{n'\} \quad \xrightarrow{\{n\} = [l]\{n'\}} \quad [\varepsilon'] = [l]^T [\varepsilon][l] \quad \Rightarrow \quad [\varepsilon] = [l][\varepsilon'][l]^T$$

Strain × displacement relations

$$[\varepsilon]=\begin{bmatrix}\varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z\end{bmatrix}=\begin{bmatrix}\varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2\\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2\\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z\end{bmatrix}$$



Engineering shear strain



Mathematical shear strain

$$\varepsilon_x=\frac{\partial u}{\partial X}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial v}{\partial X}\right)^2+\left(\frac{\partial w}{\partial X}\right)^2\right]$$

$$\varepsilon_y=\frac{\partial v}{\partial Y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2+\left(\frac{\partial v}{\partial Y}\right)^2+\left(\frac{\partial w}{\partial Y}\right)^2\right]$$

$$\varepsilon_z=\frac{\partial w}{\partial Z}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2+\left(\frac{\partial v}{\partial Z}\right)^2+\left(\frac{\partial w}{\partial Z}\right)^2\right]$$

$$\gamma_{yz}=2\varepsilon_{yz}=\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial Y}+\frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z}+\frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z}$$

$$\gamma_{xz}=2\varepsilon_{xz}=\frac{\partial u}{\partial Z}+\frac{\partial w}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial u}{\partial Z}+\frac{\partial v}{\partial X}\frac{\partial v}{\partial Z}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Z}$$

$$\gamma_{xy}=2\varepsilon_{xy}=\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}\frac{\partial v}{\partial Y}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}$$

Small strains

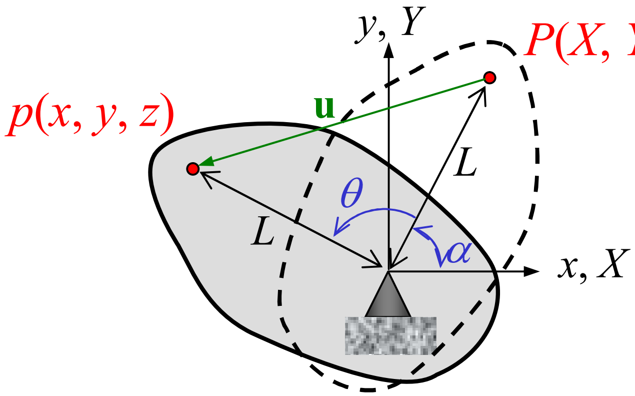
Assumption: $|\partial u_i/\partial X_j| << 1$

$$\varepsilon_x \approx \frac{\partial u}{\partial X} \qquad \varepsilon_y \approx \frac{\partial v}{\partial Y} \qquad \varepsilon_z \approx \frac{\partial w}{\partial Z}$$

$$\gamma_{yz} \approx \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z} \qquad \gamma_{xz} \approx \frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z} \qquad \gamma_{xy} \approx \frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y}$$

Although $|\partial u_i/\partial X_j| << 1$ leads to small strains nothing can be concluded about displacements. Actually there may be practical important cases where strains are small but displacements are large.

Example of pure rotation



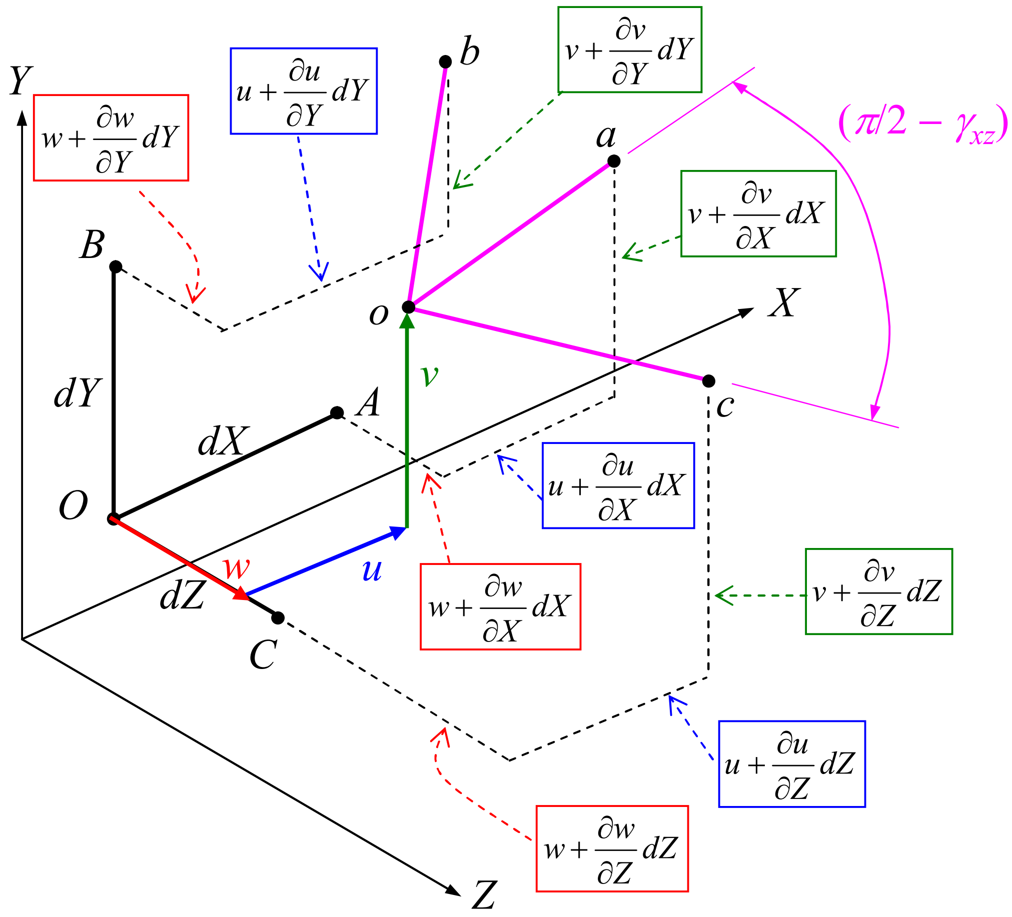
$$x = X\cos\theta - Y\sin\theta \qquad y = Y\cos\theta + X\sin\theta$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{H} = \begin{bmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[ε] = [0]. However, the small strain tensor is $\begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Geometric interpretation of strains



$A(X+dX, Y, Z)$

$B(X, Y+dY, Z)$

$C(X, Y, Z+dZ)$

$$u_A = u + (\partial u/\partial X)dX$$
$$v_A = v + (\partial v/\partial X)dX$$
$$w_A = w + (\partial w/\partial X)dX$$

$$u_B = u + (\partial u/\partial Y)dY$$
$$v_B = v + (\partial v/\partial Y)dY$$
$$w_B = w + (\partial w/\partial Y)dY$$

$$u_C = u + (\partial u/\partial Z)dZ$$
$$v_C = v + (\partial v/\partial Z)dZ$$
$$w_C = w + (\partial w/\partial Z)dZ$$

Normal strains

$$\varepsilon_x = \frac{\overline{oa} - \overline{OA}}{\overline{OA}} = \frac{\overline{oa} - dX}{dX}$$

$$(\overline{oa})^2 = \left(dX + u + \frac{\partial u}{\partial X} dX - u \right)^2 + \left(v + \frac{\partial v}{\partial X} dX - v \right)^2 + \left(w + \frac{\partial w}{\partial X} dX - w \right)^2$$

$$\overline{oa} = dX \sqrt{\left(1 + \frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2} \approx dX \left(1 + 2\frac{\partial u}{\partial X}\right)^{1/2} \approx dX \left(1 + \frac{\partial u}{\partial X}\right)$$

$$\varepsilon_x = \frac{\partial u}{\partial X} \quad , \quad \varepsilon_y = \frac{\partial v}{\partial Y} \quad , \quad \varepsilon_z = \frac{\partial w}{\partial Z}$$

Shear strains

$$\cos a\hat{o}c = \frac{(\overline{oa})^2 + (\overline{oc})^2 - (\overline{ac})^2}{2(\overline{oa})(\overline{oc})}$$

$$\overline{oa} = dX\left(1 + \frac{\partial u}{\partial X}\right) \quad , \quad \overline{oc} = dZ\left(1 + \frac{\partial w}{\partial Z}\right) \quad , \quad (\overline{ac})^2 = \left(dZ - \frac{\partial w}{\partial X} dX\right)^2 + \left(dX - \frac{\partial u}{\partial Z} dZ\right)^2$$

$$\cos a\hat{o}c \approx \frac{2(\partial w / \partial X) dXdZ + 2(\partial u / \partial Z) dXdZ}{2dXdZ} = \frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z}$$

$$\gamma_{xz} = \frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z} \quad , \quad \gamma_{xy} = \frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y} \quad , \quad \gamma_{yz} = \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z}$$

2.3. Constitutive relations

Most general stress × strain relations

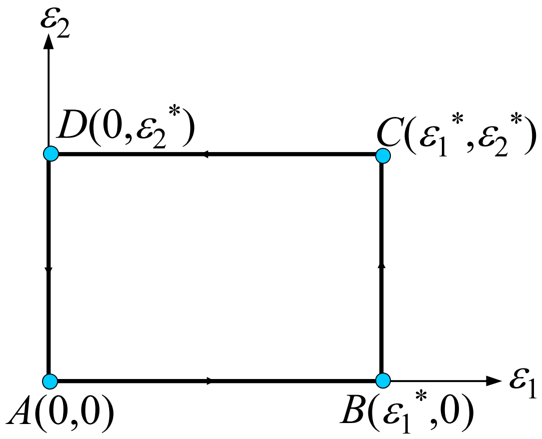
$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad \{\sigma\} = [C]\{\varepsilon\}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Most general stress × strain relations

Elastic material: $\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2$ and $\sigma_2 = C_{21}\varepsilon_1 + C_{22}\varepsilon_2$

Infinitesimal work per unit volume: $-(\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2)$



$$W_{ABCDA} = -\int_A^B (\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2) - \ldots - \int_D^A (\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2) =$$

$$-\int_0^{\varepsilon_1^*} C_{11}\varepsilon_1 d\varepsilon_1 - \int_0^{\varepsilon_2^*} (C_{21}\varepsilon_1^* + C_{22}\varepsilon_2) d\varepsilon_2 -$$

$$-\int_{\varepsilon_1^*}^0 (C_{11}\varepsilon_1 + C_{12}\varepsilon_2^*) d\varepsilon_1 - \int_{\varepsilon_2^*}^0 C_{22}\varepsilon_2 d\varepsilon_2 = (C_{12} - C_{21})\varepsilon_1^*\varepsilon_2^*$$

$W_{ADCBA} = (C_{21} - C_{12})\varepsilon_1^*\varepsilon_2^* = -W_{ABCDA} \Rightarrow$ violation of the first law of thermodynamics

Therefore, $W_{ABCDA} = 0 \Rightarrow C_{12} = C_{21}$. More generally, $C_{ij} = C_{ji}$.

Compliance matrix

$[S]=[C]^{-1}$

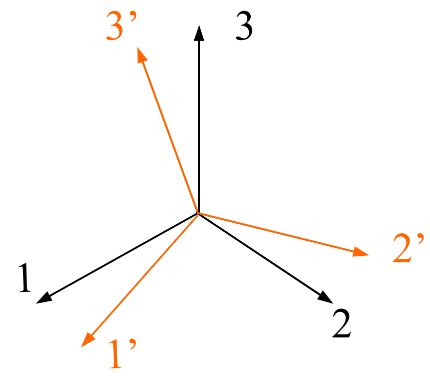
$[S]$ → compliance matrix

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}^{-1}$$

Material with one plane of symmetry

$$[\sigma'] = [l]^T [\sigma][l]$$

$$[\varepsilon'] = [l]^T [\varepsilon][l] \qquad [l] = [\{n_{1'}\} \quad \{n_{2'}\} \quad \{n_{3'}\}]$$



Assume symmetry about 12 plane. Then, if axis 3 is inverted, the constitutive relations must not change.

$$\{n_{1'}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \{n_{2'}\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \{n_{3'}\} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$

$$[\sigma'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^T \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & -\tau_{13} \\ \tau_{12} & \sigma_2 & -\tau_{23} \\ -\tau_{13} & -\tau_{23} & \sigma_3 \end{bmatrix}$$

$$[\varepsilon'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^T \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & -\varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_2 & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_3 \end{bmatrix}$$

Material with one plane of symmetry

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\C_{12}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\C_{13}&C_{23}&C_{33}&C_{34}&C_{35}&C_{36}\\C_{14}&C_{24}&C_{34}&C_{44}&C_{45}&C_{46}\\C_{15}&C_{25}&C_{35}&C_{45}&C_{55}&C_{56}\\C_{16}&C_{26}&C_{36}&C_{46}&C_{56}&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\-\tau_{23}\\-\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\C_{12}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\C_{13}&C_{23}&C_{33}&C_{34}&C_{35}&C_{36}\\C_{14}&C_{24}&C_{34}&C_{44}&C_{45}&C_{46}\\C_{15}&C_{25}&C_{35}&C_{45}&C_{55}&C_{56}\\C_{16}&C_{26}&C_{36}&C_{46}&C_{56}&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\-\gamma_{23}\\-\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&-C_{14}&-C_{15}&C_{16}\\C_{12}&C_{22}&C_{23}&-C_{24}&-C_{25}&C_{26}\\C_{13}&C_{23}&C_{33}&-C_{34}&-C_{35}&C_{36}\\-C_{14}&-C_{24}&-C_{34}&C_{44}&C_{45}&-C_{46}\\-C_{15}&-C_{25}&-C_{35}&C_{45}&C_{55}&-C_{56}\\C_{16}&C_{26}&C_{36}&-C_{46}&-C_{56}&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

$\Rightarrow$

$$C_{14}=C_{15}=C_{24}=C_{25}=C_{34}=C_{35}=C_{46}=C_{56}=0$$

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&C_{16}\\C_{12}&C_{22}&C_{23}&0&0&C_{26}\\C_{13}&C_{23}&C_{33}&0&0&C_{36}\\0&0&0&C_{44}&C_{45}&0\\0&0&0&C_{45}&C_{55}&0\\C_{16}&C_{26}&C_{36}&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

Material with two planes of symmetry

Assume symmetry about 13 plane. Then, if axis 2 is inverted, the constitutive relations must not change.

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&-C_{14}&C_{15}&-C_{16}\\C_{12}&C_{22}&C_{23}&-C_{24}&C_{25}&-C_{26}\\C_{13}&C_{23}&C_{33}&-C_{34}&C_{35}&-C_{36}\\-C_{14}&-C_{24}&-C_{34}&C_{44}&-C_{45}&C_{46}\\C_{15}&C_{25}&C_{35}&-C_{45}&C_{55}&-C_{56}\\-C_{16}&-C_{26}&-C_{36}&C_{46}&-C_{56}&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

$$\Rightarrow\quad C_{14}=C_{16}=C_{24}=C_{26}=C_{34}=C_{36}=C_{45}=C_{56}=0$$

Symmetry about 12 and 13 planes: $\Rightarrow$

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\C_{12}&C_{22}&C_{23}&0&0&0\\C_{13}&C_{23}&C_{33}&0&0&0\\0&0&0&C_{44}&0&0\\0&0&0&0&C_{55}&0\\0&0&0&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

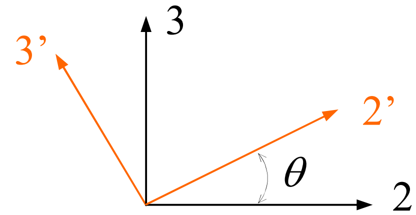
Symmetry about two planes implies symmetry about three planes!

Transversely isotropic material

Assume the material is transversely isotropic on plane 23. Then rotations of arbitrary angles about axis 1 must not change the constitutive relations.

$$\{n_{1'}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \{n_{2'}\} = \begin{Bmatrix} 0 \\ c \\ s \end{Bmatrix} \quad \{n_{3'}\} = \begin{Bmatrix} 0 \\ -s \\ c \end{Bmatrix}$$

$$c = \cos\theta$$
$$s = \sin\theta$$



$$[\sigma'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}^T \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} = \begin{bmatrix} \sigma_1 & c\tau_{12} + s\tau_{13} & c\tau_{13} - s\tau_{12} \\ c\tau_{12} + s\tau_{13} & c^2\sigma_2 + 2cs\tau_{23} + s^2\sigma_3 & (c^2 - s^2)\tau_{23} + sc(\sigma_3 - \sigma_2) \\ c\tau_{13} - s\tau_{12} & (c^2 - s^2)\tau_{23} + sc(\sigma_3 - \sigma_2) & s^2\sigma_2 - 2cs\tau_{23} + c^2\sigma_3 \end{bmatrix}$$

$$[\varepsilon'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}^T \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & c\varepsilon_{12} + s\varepsilon_{13} & c\varepsilon_{13} - s\varepsilon_{12} \\ c\varepsilon_{12} + s\varepsilon_{13} & c^2\varepsilon_2 + 2cs\varepsilon_{23} + s^2\varepsilon_3 & (c^2 - s^2)\varepsilon_{23} + sc(\varepsilon_3 - \varepsilon_2) \\ c\varepsilon_{13} - s\varepsilon_{12} & (c^2 - s^2)\varepsilon_{23} + sc(\varepsilon_3 - \varepsilon_2) & s^2\varepsilon_2 - 2cs\varepsilon_{23} + c^2\varepsilon_3 \end{bmatrix}$$

Transversely isotropic material with 3 planes of symmetry

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{23}\\\tau_{13}\\\tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\C_{12}&C_{22}&C_{23}&0&0&0\\C_{13}&C_{23}&C_{33}&0&0&0\\0&0&0&C_{44}&0&0\\0&0&0&0&C_{55}&0\\0&0&0&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{23}\\\gamma_{13}\\\gamma_{12}\end{Bmatrix}$$

$$c\tau_{12}+s\tau_{13}=C_{66}(c\gamma_{12}+s\gamma_{13}) \qquad \tau_{12}=C_{66}\gamma_{12}$$
$$c\tau_{13}-s\tau_{12}=C_{55}(c\gamma_{13}-s\gamma_{12}) \qquad \tau_{13}=C_{55}\gamma_{13}$$

$\Rightarrow C_{55}=C_{66}$

$$\sigma_1=C_{11}\varepsilon_1+C_{12}(c^2\varepsilon_2+s^2\varepsilon_3+\gamma_{23}cs)+C_{13}(c^2\varepsilon_3+s^2\varepsilon_2-\gamma_{23}cs)$$
$$\sigma_1=C_{11}\varepsilon_1+C_{12}\varepsilon_2+C_{13}\varepsilon_3$$

$\Rightarrow C_{12}=C_{13}$

$$c^2\sigma_2+2sc\tau_{23}+s^2\sigma_3=C_{12}\varepsilon_1+C_{22}(c^2\varepsilon_2+s^2\varepsilon_3+\gamma_{23}cs)+C_{23}(c^2\varepsilon_3+s^2\varepsilon_2-\gamma_{23}cs)$$
$$\sigma_2=C_{12}\varepsilon_1+C_{22}\varepsilon_2+C_{23}\varepsilon_3$$
$$c^2\sigma_3-2sc\tau_{23}+s^2\sigma_2=C_{13}\varepsilon_1+C_{23}(c^2\varepsilon_2+s^2\varepsilon_3+\gamma_{23}cs)+C_{33}(c^2\varepsilon_3+s^2\varepsilon_2-\gamma_{23}cs)$$
$$\sigma_3=C_{13}\varepsilon_1+C_{23}\varepsilon_2+C_{33}\varepsilon_3$$

$\Rightarrow C_{22}=C_{33}$

$$(c^2-s^2)\tau_{23}+sc(\sigma_3-\sigma_2)=C_{44}[(c^2-s^2)\gamma_{23}+2sc(\varepsilon_3-\varepsilon_2)]$$
$$\tau_{23}=C_{44}\gamma_{23}$$

$\Rightarrow C_{44}=(C_{22}-C_{23})/2$

Transversely isotropic material with 3 planes of symmetry

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{22}-C_{23})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Only five constants are required: C_{11}, C_{22}, C_{12}, C_{23} and C_{55}.

Completely isotropic material

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Only two constants are required: C_{11} and C_{12}.

Compliance matrices

One plane of symmetry

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Two or three planes of symmetry

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Transversely isotropic

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Completely isotropic

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Physical significance of anisotropic stress × strain relations

extension-extension coupling

extension

shear-extension coupling

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

shear-shear coupling

shear

2.4. Stiffnesses, compliances and engineering constants

Engineering constants

Generalized Young moduli, Poisson ratios and shear moduli

Simple tests used to measure engineering constants

More easily defined in terms of compliances

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

E_1, E_2, E_3: Young moduli

ν_{12}, ν_{13}, ν_{23}: Poisson ratios

G_{23}, G_{13}, G_{12}: shear moduli

$\nu_{ij} = -\varepsilon_j / \varepsilon_i$

$$S_{ij} = S_{ji} \Rightarrow \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$$

Engineering constants

Determination of engineering constants. Perform six tests:

1) Load σ_1 and measure ε_1, ε_2, ε_3

2) Load σ_2 and measure ε_1, ε_2, ε_3

3) Load σ_3 and measure ε_1, ε_2, ε_3

4) Load τ_{23} and measure γ_{23}

5) Load τ_{13} and measure γ_{13}

6) Load τ_{12} and measure γ_{12}

Tests 1, 2 and 3 provide Young moduli and Poisson ratios

Tests 4, 5 and 6 provide shear moduli

Compliance matrices

Two or three planes of symmetry

$$C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{S} \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S} \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}$$

$$C_{22} = \frac{S_{33}S_{11} - S_{13}^2}{S} \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S} \quad C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{S}$$

$$C_{44} = \frac{1}{S_{44}} \quad C_{55} = \frac{1}{S_{55}} \quad C_{66} = \frac{1}{S_{66}}$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}$$

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} \qquad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta}$$

$$C_{12} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta} \quad C_{23} = \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2\Delta}$$

$$C_{13} = \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta} \qquad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta}$$

$$C_{44} = G_{23} \quad C_{55} = G_{13} \quad C_{66} = G_{12}$$

$$\Delta = \frac{1-\nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

Restrictions on engineering constraints

Isotropic materials

$$G = \frac{E}{2(1+\nu)} \quad \Rightarrow \quad \nu > -1 \qquad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{3p(1-2\nu)}{E} \quad \Rightarrow \quad \nu < \frac{1}{2}$$

$$\boxed{-1 < \nu < \frac{1}{2}}$$

Orthotropic materials

$$S_{11}, S_{22}, S_{33}, S_{44}, S_{55}, S_{66} > 0 \quad \Rightarrow \quad E_1, E_2, E_3, G_{23}, G_{13}, G_{12} > 0$$

$$C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66} > 0 \quad \Rightarrow \quad \begin{cases} 1-\nu_{23}\nu_{32} > 0, \quad 1-\nu_{13}\nu_{31} > 0, \quad 1-\nu_{21}\nu_{12} > 0 \\ \bar{\Delta} = 1-\nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} > 0 \end{cases}$$

$$|S_{23}| < \sqrt{S_{22}S_{33}} \quad , \quad |S_{13}| < \sqrt{S_{11}S_{33}} \quad , \quad |S_{12}| < \sqrt{S_{11}S_{22}}$$

$$|\nu_{21}| < \sqrt{\frac{E_2}{E_1}} \quad |\nu_{32}| < \sqrt{\frac{E_3}{E_2}} \quad |\nu_{13}| < \sqrt{\frac{E_1}{E_3}}$$

$$|\nu_{12}| < \sqrt{\frac{E_1}{E_2}} \quad |\nu_{23}| < \sqrt{\frac{E_2}{E_3}} \quad |\nu_{31}| < \sqrt{\frac{E_3}{E_1}}$$

$$\nu_{21}\nu_{32}\nu_{13} < \frac{1-\nu_{21}^2\frac{E_1}{E_2} - \nu_{32}^2\frac{E_2}{E_3} - \nu_{13}^2\frac{E_3}{E_1}}{2} < \frac{1}{2}$$

2.5. Stress × strain relations for laminae

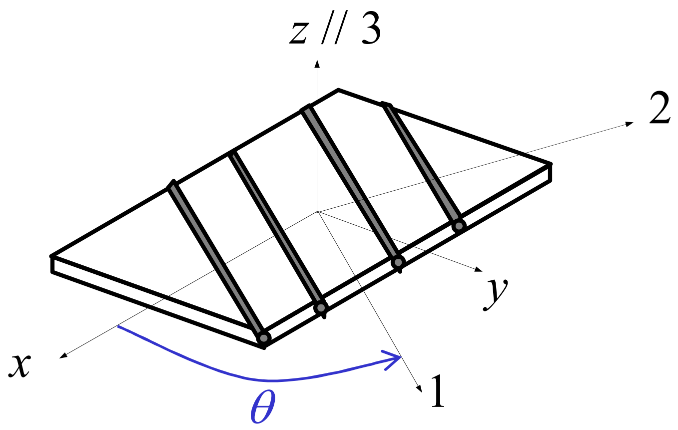
Rotation about axis 3 (or *z* axis)

Structural axes x, y, z

Material axes 1, 2, 3

Axis 1 aligned with fiber direction

$c = \cos\theta,\ s = \sin\theta$



$$[\sigma_{123}] = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c^2\sigma_x + s^2\sigma_y + 2cs\tau_{xy} & (c^2 - s^2)\tau_{xy} + cs(\sigma_y - \sigma_x) & c\tau_{xz} + s\tau_{yz} \\ (c^2 - s^2)\tau_{xy} + cs(\sigma_y - \sigma_x) & s^2\sigma_x + c^2\sigma_y - 2cs\tau_{xy} & c\tau_{yz} - s\tau_{xz} \\ c\tau_{xz} + s\tau_{yz} & c\tau_{yz} - s\tau_{xz} & \sigma_z \end{bmatrix}$$

$$[\varepsilon_{123}] = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c^2\varepsilon_x + s^2\varepsilon_y + 2cs\varepsilon_{xy} & (c^2 - s^2)\varepsilon_{xy} + cs(\varepsilon_y - \varepsilon_x) & c\varepsilon_{xz} + s\varepsilon_{yz} \\ (c^2 - s^2)\varepsilon_{xy} + cs(\varepsilon_y - \varepsilon_x) & s^2\varepsilon_x + c^2\varepsilon_y - 2cs\varepsilon_{xy} & c\varepsilon_{yz} - s\varepsilon_{xz} \\ c\varepsilon_{xz} + s\varepsilon_{yz} & c\varepsilon_{yz} - s\varepsilon_{xz} & \varepsilon_z \end{bmatrix}$$

Vector notation

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Two or three planes of symmetry

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Vector notation

$$\begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Vector notation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

$$\overline{C}_{11} = C_{11}c^4 + C_{22}s^4 + 2c^2s^2C_{12} + 4c^2s^2C_{66}$$

$$\overline{C}_{12} = (C_{11} + C_{22})c^2s^2 + C_{12}(c^4 + s^4) - 4c^2s^2C_{66}$$

$$\overline{C}_{13} = C_{13}c^2 + C_{23}s^2$$

$$\overline{C}_{14} = \overline{C}_{15} = 0$$

$$\overline{C}_{16} = (C_{11} - C_{12})c^3s + (C_{12} - C_{22})cs^3 - 2cs(c^2 - s^2)C_{66}$$

$$\overline{C}_{22} = C_{11}s^4 + C_{22}c^4 + 2c^2s^2C_{12} + 4c^2s^2C_{66}$$

$$\overline{C}_{23} = C_{13}s^2 + C_{23}c^2$$

$$\overline{C}_{24} = \overline{C}_{25} = 0$$

$$\overline{C}_{26} = (C_{11} - C_{12})cs^3 + (C_{12} - C_{22})c^3s + 2cs(c^2 - s^2)C_{66}$$

$$\overline{C}_{33} = C_{33}$$

$$\overline{C}_{34} = \overline{C}_{35} = 0$$

$$\overline{C}_{36} = (C_{13} - C_{23})cs$$

$$\overline{C}_{44} = C_{44}c^2 + C_{55}s^2$$

$$\overline{C}_{45} = (C_{55} - C_{44})cs$$

$$\overline{C}_{46} = 0$$

$$\overline{C}_{55} = C_{44}s^2 + C_{55}c^2$$

$$\overline{C}_{56} = 0$$

$$\overline{C}_{66} = (C_{11} + C_{22} - 2C_{12})c^2s^2 + C_{66}(c^2 - s^2)^2$$

Lamina under plane stress state

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$Q_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} \quad , \quad Q_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}}$$

$$Q_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} \quad , \quad Q_{66} = C_{66}$$

Introducing engineering constants:

$$S_{11} = \frac{1}{E_1} \qquad S_{22} = \frac{1}{E_2} \qquad S_{12} = -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1} \qquad S_{66} = \frac{1}{G_{12}}$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

Lamina under plane stress state

Stress and strain transformations:

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}c^2 & s^2 & 2cs\\ s^2 & c^2 & -2cs\\ -cs & cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}\qquad \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\begin{bmatrix}c^2 & s^2 & cs\\ s^2 & c^2 & -cs\\ -2cs & 2cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{bmatrix}c^2 & s^2 & 2cs\\ s^2 & c^2 & -2cs\\ -cs & cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{bmatrix}c^2 & s^2 & cs\\ s^2 & c^2 & -cs\\ -2cs & 2cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}c^2 & s^2 & -2cs\\ s^2 & c^2 & 2cs\\ cs & -cs & c^2-s^2\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{bmatrix}c^2 & s^2 & cs\\ s^2 & c^2 & -cs\\ -2cs & 2cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{xx} & Q_{xy} & Q_{xs}\\ Q_{xy} & Q_{yy} & Q_{ys}\\ Q_{xs} & Q_{ys} & Q_{ss}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$Q_{xx}=Q_{11}c^4+Q_{22}s^4+2(Q_{12}+2Q_{66})c^2s^2$$
$$Q_{yy}=Q_{11}s^4+Q_{22}c^4+2(Q_{12}+2Q_{66})c^2s^2$$
$$Q_{xy}=(Q_{11}+Q_{22}-4Q_{66})c^2s^2+Q_{12}(c^4+s^4)$$
$$Q_{ss}=(Q_{11}+Q_{22}-2Q_{12})c^2s^2+Q_{66}(c^2-s^2)^2$$

Even functions of θ

$$Q_{xs}=(Q_{11}-Q_{12})c^3s+(Q_{12}-Q_{22})cs^3-2Q_{66}cs(c^2-s^2)$$
$$Q_{ys}=(Q_{11}-Q_{12})cs^3+(Q_{12}-Q_{22})c^3s+2Q_{66}cs(c^2-s^2)$$

Odd functions of θ

Lamina under plane stress state

Relations in terms of compliances

$$\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2-s^2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{xy} & S_{yy} & S_{ys} \\ S_{xs} & S_{ys} & S_{ss} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$S_{xx} = S_{11}c^4 + S_{22}s^4 + (2S_{12} + S_{66})c^2s^2$$

$$S_{yy} = S_{11}s^4 + S_{22}c^4 + (2S_{12} + S_{66})c^2s^2$$

$$S_{xy} = (S_{11} + S_{22} - S_{66})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S_{ss} = 4(S_{11} + S_{22} - 2S_{12})c^2s^2 + S_{66}(c^2 - s^2)^2$$

$$S_{xs} = 2(S_{11} - S_{12})c^3s + 2(S_{12} - S_{22})cs^3 - S_{66}cs(c^2 - s^2)$$

$$S_{ys} = 2(S_{11} - S_{12})cs^3 + 2(S_{12} - S_{22})c^3s + S_{66}cs(c^2 - s^2)$$

2.6. Stress × strain relations in terms of engineering constants

Transformed engineering constants

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_s\end{Bmatrix}=\begin{bmatrix}S_{xx} & S_{xy} & S_{xs}\\ S_{xy} & S_{yy} & S_{ys}\\ S_{xs} & S_{ys} & S_{ss}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_s\end{Bmatrix}=\begin{bmatrix}\dfrac{1}{E_x} & -\dfrac{\nu_{xy}}{E_x} & \dfrac{\eta_{xs}}{E_x}\\ -\dfrac{\nu_{yx}}{E_y} & \dfrac{1}{E_y} & \dfrac{\eta_{ys}}{E_y}\\ \dfrac{\eta_{sx}}{G_{xy}} & \dfrac{\eta_{sy}}{G_{xy}} & \dfrac{1}{G_{xy}}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_s\end{Bmatrix}$$

η_{xs}, η_{ys} , η_{sx} , η_{sy} $\rightarrow$ coefficients of mutual influence

$$\frac{1}{E_x}=\frac{1}{E_1}c^4+\frac{1}{E_2}s^4+\left(\frac{1}{G_{12}}-\frac{2\nu_{12}}{E_1}\right)c^2s^2$$

$$\frac{1}{E_y}=\frac{1}{E_1}s^4+\frac{1}{E_2}c^4+\left(\frac{1}{G_{12}}-\frac{2\nu_{12}}{E_1}\right)c^2s^2$$

$$\frac{\nu_{xy}}{E_x}=\frac{\nu_{12}}{E_1}(c^4+s^4)-\left(\frac{1}{E_1}+\frac{1}{E_2}-\frac{1}{G_{12}}\right)c^2s^2$$

$$\frac{1}{G_{xy}}=4\left(\frac{1}{E_1}+\frac{1}{E_2}+\frac{2\nu_{12}}{E_1}\right)c^2s^2+\frac{1}{G_{12}}(c^2-s^2)^2$$

$$\frac{\eta_{xs}}{E_x}=2\left(\frac{1}{E_1}+\frac{\nu_{12}}{E_1}\right)c^3s-2\left(\frac{\nu_{21}}{E_2}+\frac{1}{E_2}\right)cs^3-\frac{1}{G_{12}}cs(c^2-s^2)$$

$$\frac{\eta_{ys}}{E_y}=2\left(\frac{1}{E_1}+\frac{\nu_{12}}{E_1}\right)cs^3-2\left(\frac{\nu_{21}}{E_2}+\frac{1}{E_2}\right)c^3s+\frac{1}{G_{12}}cs(c^2-s^2)$$

Transformed engineering constants

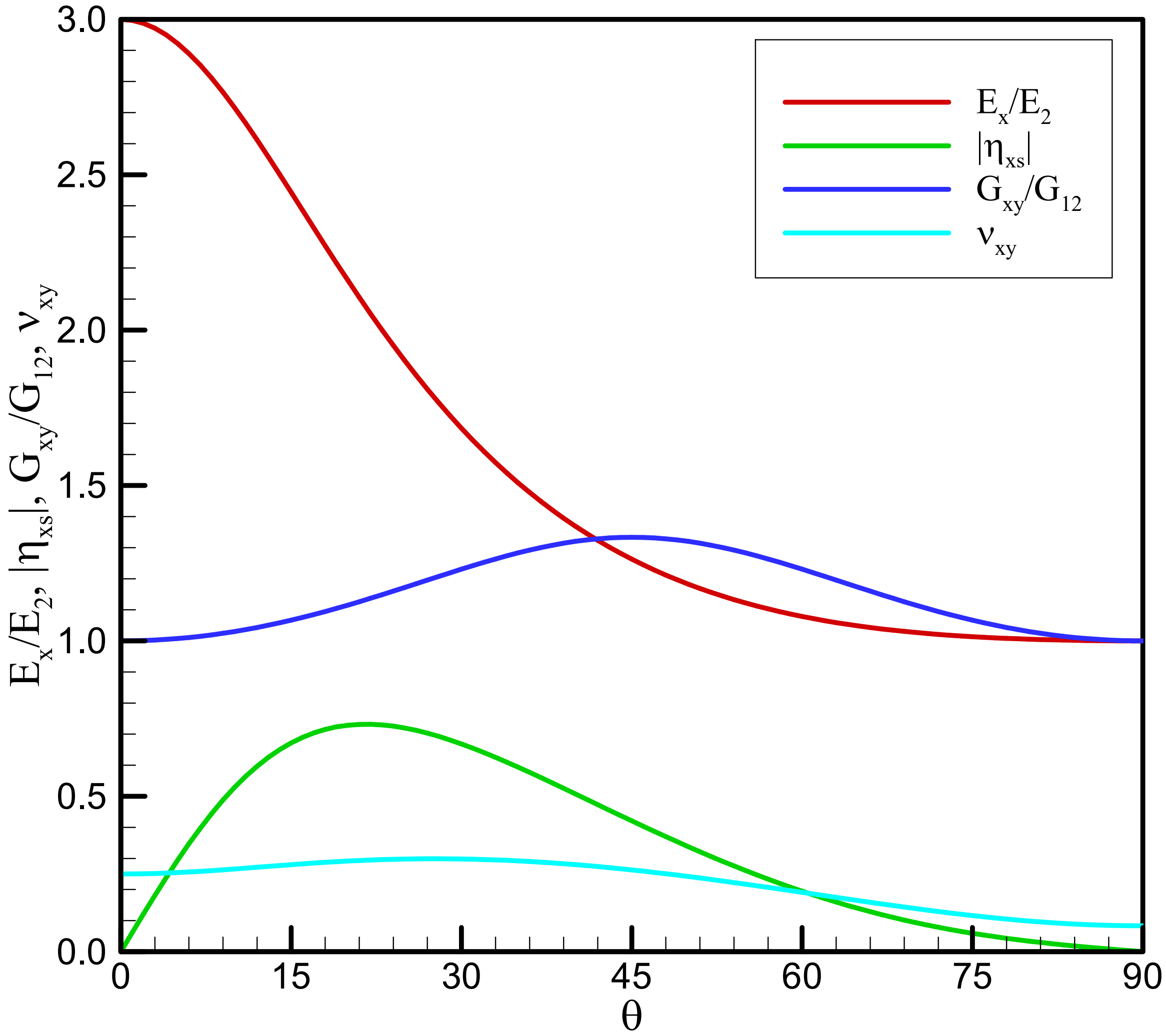
Glass-epoxy

$E_1 = 7.8 \times 10^6$ psi

$E_2 = 2.6 \times 10^6$ psi

$\nu_{12} = 0.25$

$G_{12} = 1.3 \times 10^6$ psi



Transformed engineering constants

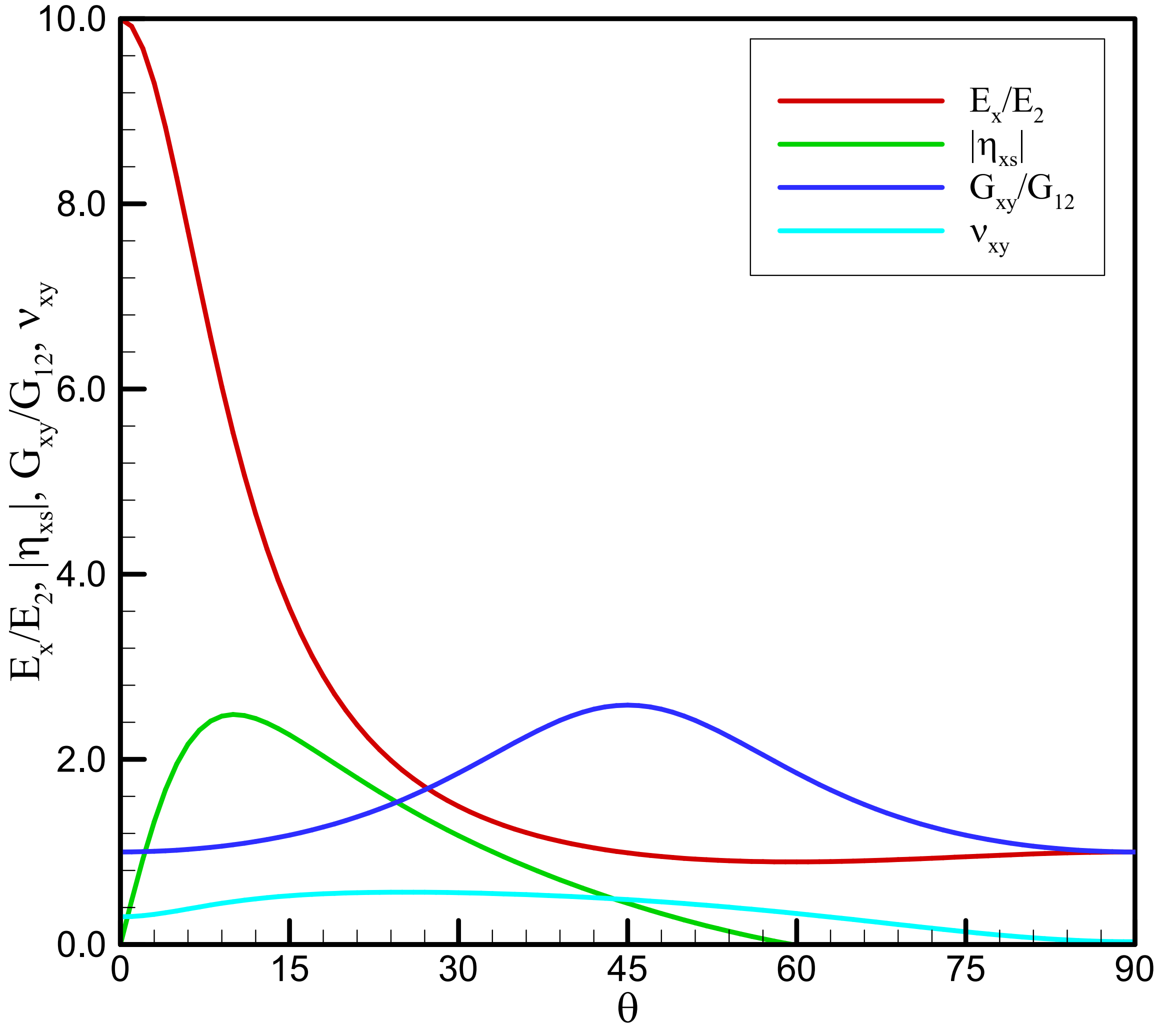
Boron-epoxy

$E_1 = 30.0\times10^6$ psi

$E_2 = 3.0\times10^6$ psi

$\nu_{12} = 0.3$

$G_{12} = 1.0\times10^6$ psi



2.7. Invariant properties

Trigonometric identities

$$c^4 = (3 + 4\cos 2\theta + \cos 4\theta)/8$$
$$c^3 s = (2\sin 2\theta + \sin 4\theta)/8$$
$$c^2 s^2 = (1 - \cos 4\theta)/8$$
$$cs^3 = (2\sin 2\theta - \sin 4\theta)/8$$
$$s^4 = (3 - 4\cos 2\theta + \cos 4\theta)/8$$

$$U_1 = (3C_{11} + 3C_{22} + 2C_{12} + 4C_{66})/8$$
$$U_2 = (C_{11} - C_{22})/2$$
$$U_3 = (C_{11} + C_{22} - 2C_{12} - 4C_{66})/8$$
$$U_4 = (C_{11} + C_{22} + 6C_{12} - 4C_{66})/8$$
$$U_5 = (C_{44} + C_{55})/2$$
$$U_6 = (C_{55} - C_{44})/2$$

$$\overline{C}_{11} = C_{11}c^4 + C_{22}s^4 + 2(C_{12} + 2C_{66})c^2 s^2 = U_1 + U_2\cos 2\theta + U_3\cos 4\theta$$
$$\overline{C}_{22} = C_{11}s^4 + C_{22}c^4 + 2(C_{12} + 2C_{66})c^2 s^2 = U_1 - U_2\cos 2\theta + U_3\cos 4\theta$$
$$\overline{C}_{12} = (C_{11} + C_{22} - 4C_{66})c^2 s^2 + C_{12}(c^4 + s^4) = U_4 - U_3\cos 4\theta$$
$$\overline{C}_{66} = (C_{11} + C_{22} - 2C_{12})c^2 s^2 + C_{66}(c^2 - s^2)^2 = (U_1 - U_4)/2 - U_3\cos 4\theta$$
$$\overline{C}_{16} = (C_{11} - C_{12})c^3 s + (C_{12} - C_{22})cs^3 - 2C_{66}cs(c^2 - s^2) = U_2\sin 2\theta/2 + U_3\sin 4\theta$$
$$\overline{C}_{26} = (C_{11} - C_{12})cs^3 + (C_{12} - C_{22})c^3 s + 2C_{66}cs(c^2 - s^2) = U_2\sin 2\theta/2 - U_3\sin 4\theta$$
$$\overline{C}_{44} = C_{44}c^2 + C_{55}s^2 = U_5 + U_6\cos 2\theta$$
$$\overline{C}_{45} = (C_{55} - C_{44})cs = U_6\sin 2\theta$$
$$\overline{C}_{55} = C_{44}s^2 + C_{55}c^2 = U_5 - U_6\cos 2\theta$$

Invariant properties

Clear way of separating contributions in the calculation of $\overline{C}_{ij}$

Two frequencies are involved: 2θ and 4θ

More useful in the study of laminates

Lamination parameters can be defined using invariants

2.8. Strength of orthotropic lamina

Strength concepts

In isotropic materials the important aspect for strength prediction is intensity of stress/strain irrespective of direction. Determination of principal stresses and strains is the objective

In composites direction of stresses and strains is paramount to determine strength

Orthotropy means that the axes of principal stresses and principal strains do not coincide

The highest stress might not be the stress driving the design

Fundamental strengths

Lamina stressed in its own plane.

Example: $X = 350$ MPa, $Y = 7$ MPa, $S = 14$ MPa, $\sigma_1 = 315$ MPa, $\sigma_2 = 14$ MPa, $\tau_{12} = 7$ MPa $\Rightarrow \sigma_1 < X$ but $\sigma_2 > Y$.

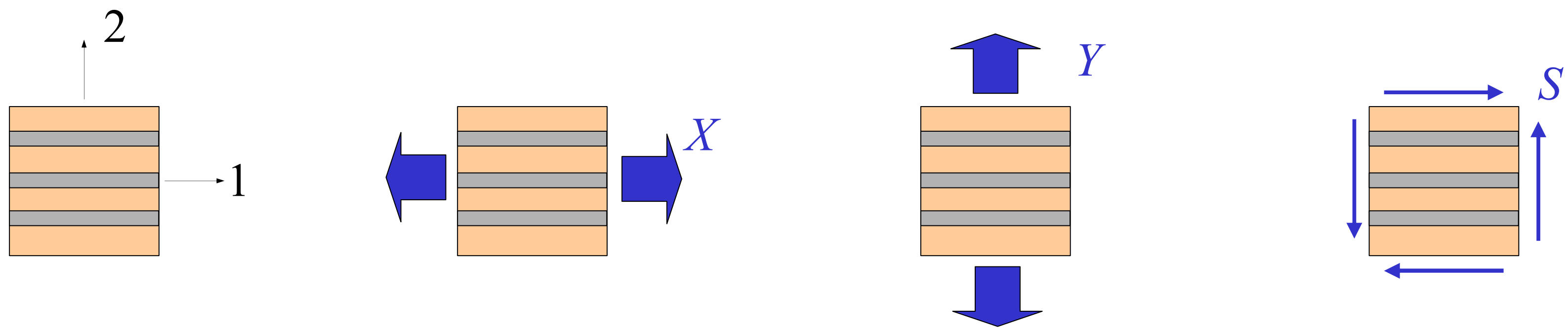
X_t = axial or longitudinal strength in tension

X_c = axial or longitudinal strength in compression

Y_t = transverse strength in tension

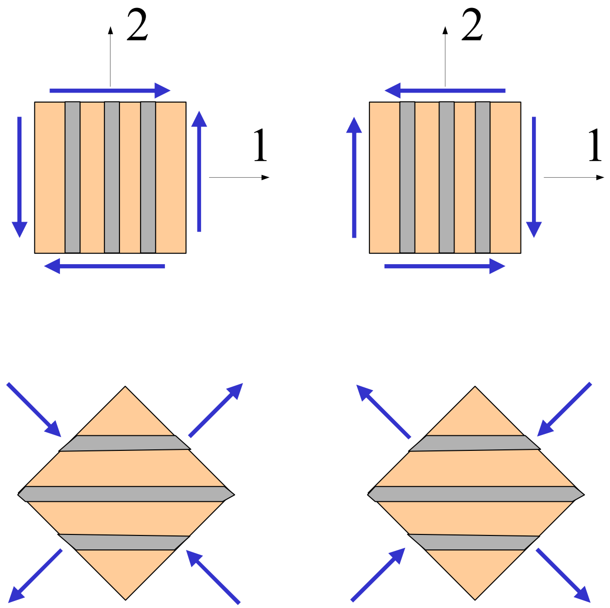
Y_c = transverse strength in compression

S = shear strength

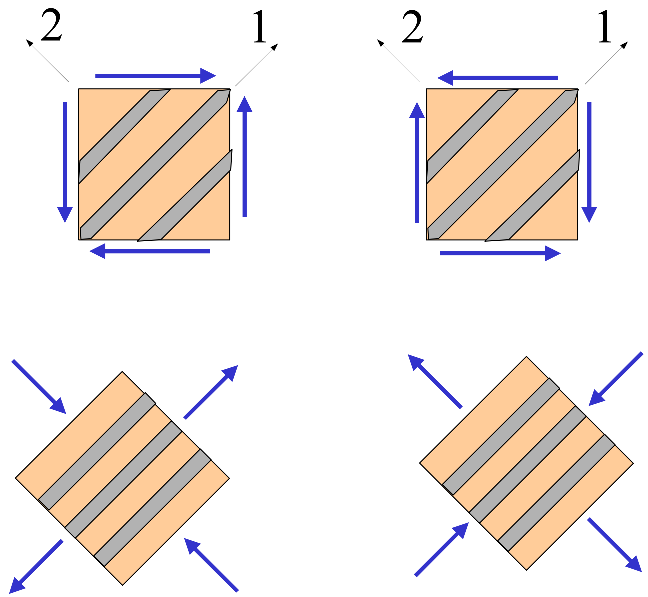


Shear strengths

Shear along material axes

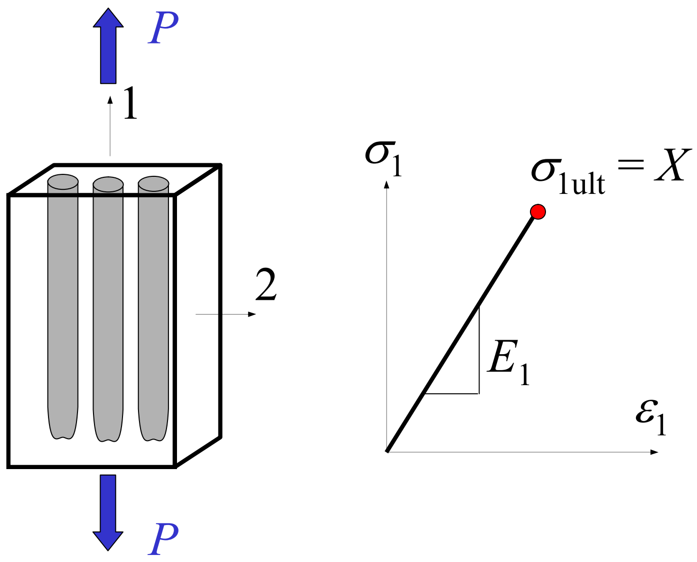


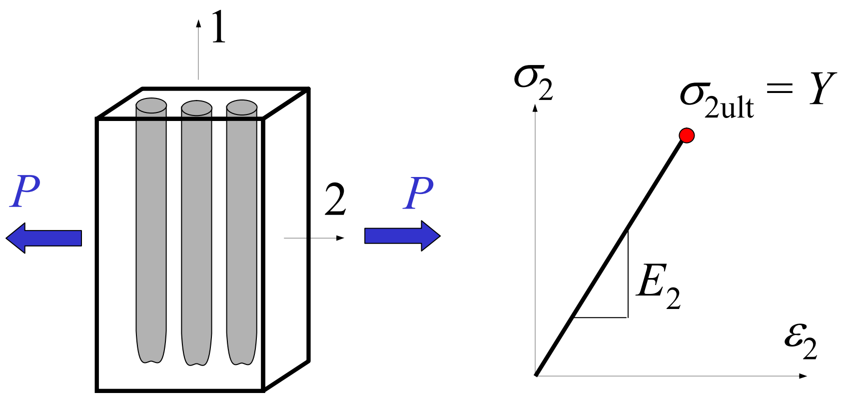
Shear at 45°



Experimental determination of normal strengths

Normal strengths are relatively easy to be determined through experiments adapted from metal (ASTM D 638).





$$\sigma_1 = \frac{P}{A} \qquad E_1 = \frac{\sigma_1}{\varepsilon_1}$$

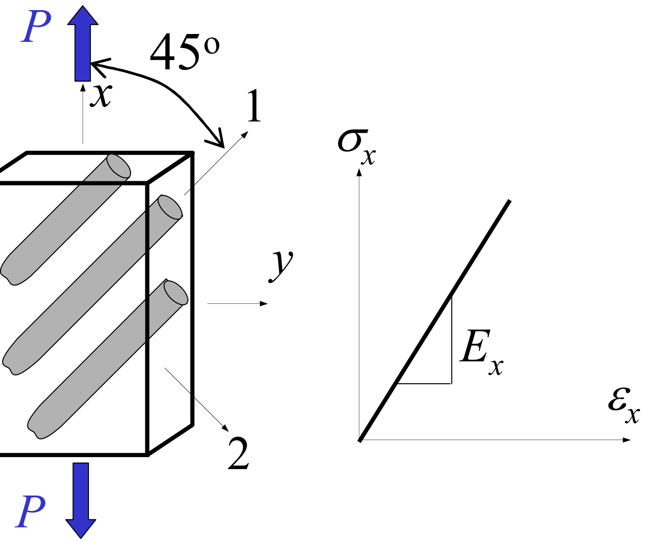
$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \qquad X = \frac{P_{ult}}{A}$$

$$\sigma_2 = \frac{P}{A} \qquad E_2 = \frac{\sigma_2}{\varepsilon_2}$$

$$\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} \qquad Y = \frac{P_{ult}}{A}$$

Experimental determination of shear strength

Shear strengths are much harder to determine experimentally

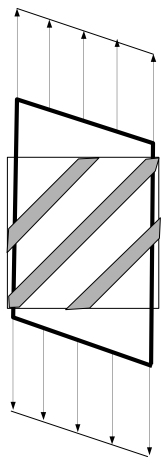


$$\frac{1}{E_x} = \frac{1}{4}\left(\frac{1}{E_1} - 2\frac{\nu_{12}}{E_1} + \frac{1}{G_{12}} + \frac{1}{E_2}\right)$$

$$G_{12} = \left(\frac{4}{E_x} - \frac{1}{E_1} - \frac{1}{E_2} + 2\frac{\nu_{12}}{E_1}\right)^{-1}$$

Strengths do not transform like stiffnesses!

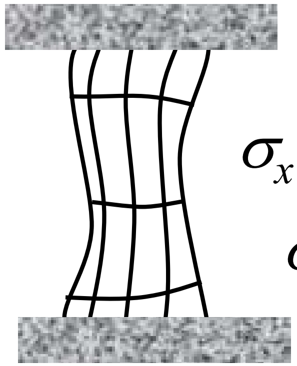
Experimental determination of shear strength



No end effect

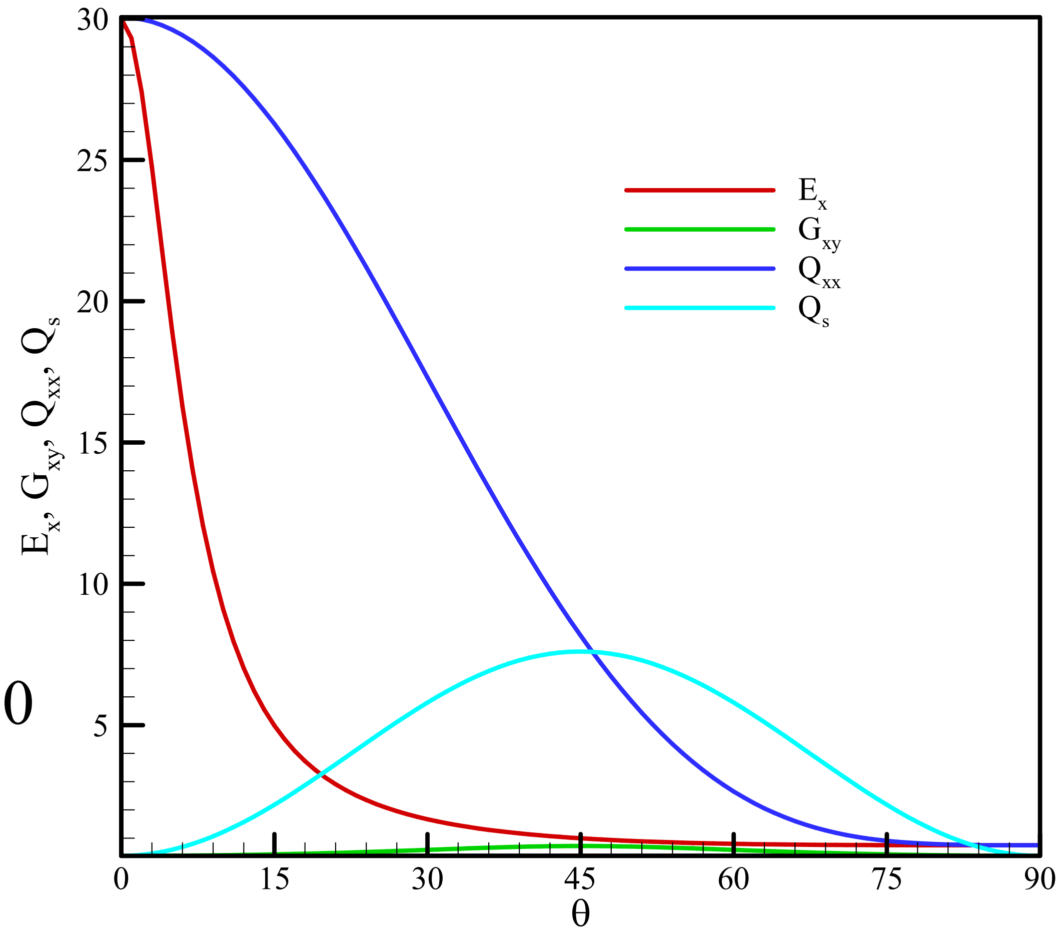
$$\sigma_x = E_x \varepsilon_x$$

Restrained ends



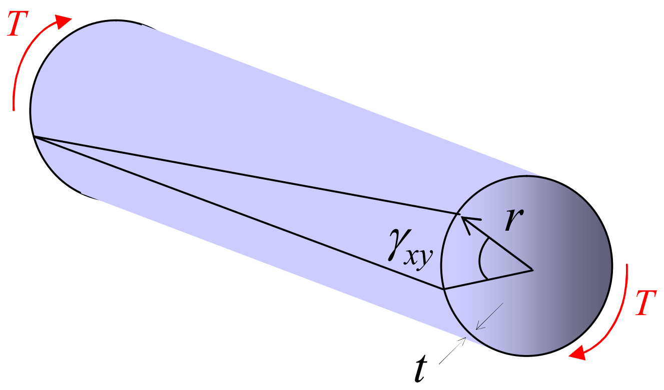
$$\sigma_x \neq 0,\ \varepsilon_y = \gamma_{xy} = 0$$

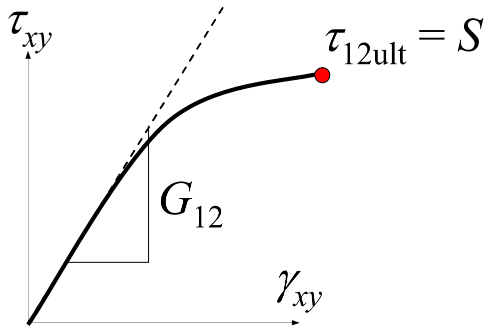
$$\sigma_x = Q_{xx} \varepsilon_x$$



Experimental determination of shear strength

Torsion tube best





$$\tau_{12} = \frac{T}{2\pi r^2 t} \qquad S = \frac{T_{\text{ult}}}{2\pi r^2 t} \qquad G_{12} = \frac{\tau_{12}}{\gamma_{12}}$$

2.9. Biaxial strength criteria for orthotropic lamina

General comments on failure criteria

Measurements of strengths are based on uni-axial tests

In practice bi- or tri-axial states exist

Several failure mechanisms exist so using transformation of tensor of strengths is not viable

Phenomenological failure criteria are proposed. They are not based on sound theory and consist rather of empirical approach

Failure envelopes are constructed by curve fitting. The envelopes merely represent the stress level at which failure occurs

Curve fitting hides the actual failure mechanism taking place

"Most failure criteria are meaningless curves passed through unrelated data points". J. Hart-Smith

Mechanisms of failure

Longitudinal traction → fiber rupture

Longitudinal compression → microbuckling and kinking

Transverse traction

Transverse compression } cracks parallel to the fibers

Shear } weak interface fiber/matrix

Biaxial strength criteria

1) Maximum stress failure criterion

2) Maximum strain failure criterion

3) Tsai-Hill failure criterion

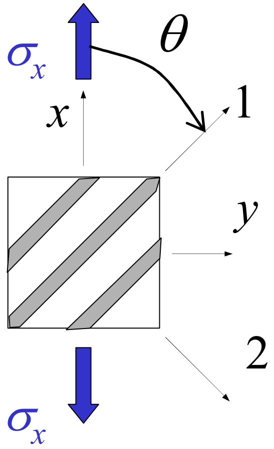
4) Hoffman failure criterion

5) Tsai-Wu tensor failure criterion

Biaxial strength criteria

Biaxial stresses generated by off-axis loading

Mixed tension and compression cannot be induced by σ_x



$$\sigma_1 = \sigma_x \cos^2\theta$$

$$\sigma_2 = \sigma_x \sin^2\theta$$

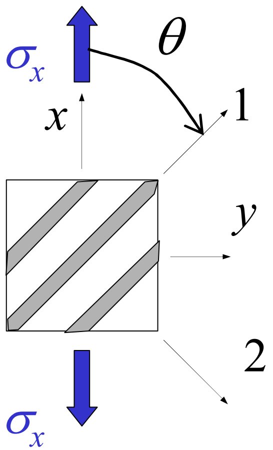
$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$

1) Maximum stress failure criterion

Tensile stresses: $\sigma_1 < X_t$, $\sigma_2 < Y_t$

Compressive stresses: $\sigma_1 > X_c$, $\sigma_2 > Y_c$

Shear: $|\tau| < S$



$$\sigma_1 = \sigma_x \cos^2\theta$$

$$\sigma_2 = \sigma_x \sin^2\theta$$

$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$

$$\Rightarrow$$

$$\frac{X_c}{\cos^2\theta} < \sigma_x < \frac{X_t}{\cos^2\theta}$$

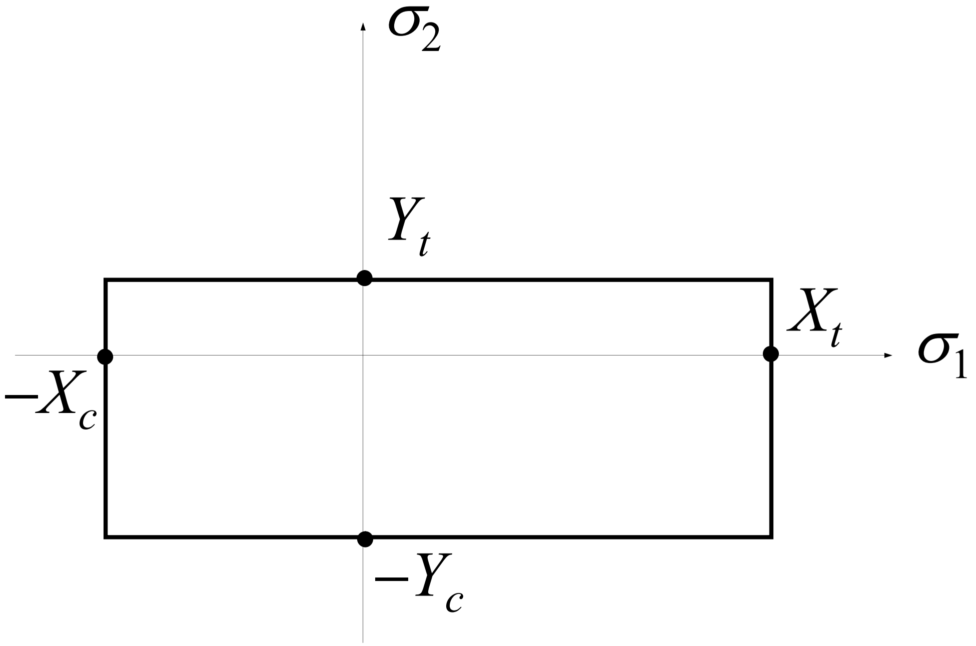
$$\frac{Y_c}{\sin^2\theta} < \sigma_x < \frac{Y_t}{\sin^2\theta}$$

$$|\sigma_x| < \left|\frac{S}{\sin\theta\cos\theta}\right|$$

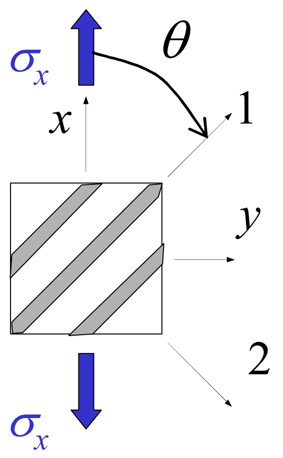
The maximum stress criterion indicates the mechanism behind failure. It suffices to check which inequality is violated.

1) Maximum stress failure criterion

Failure envelope



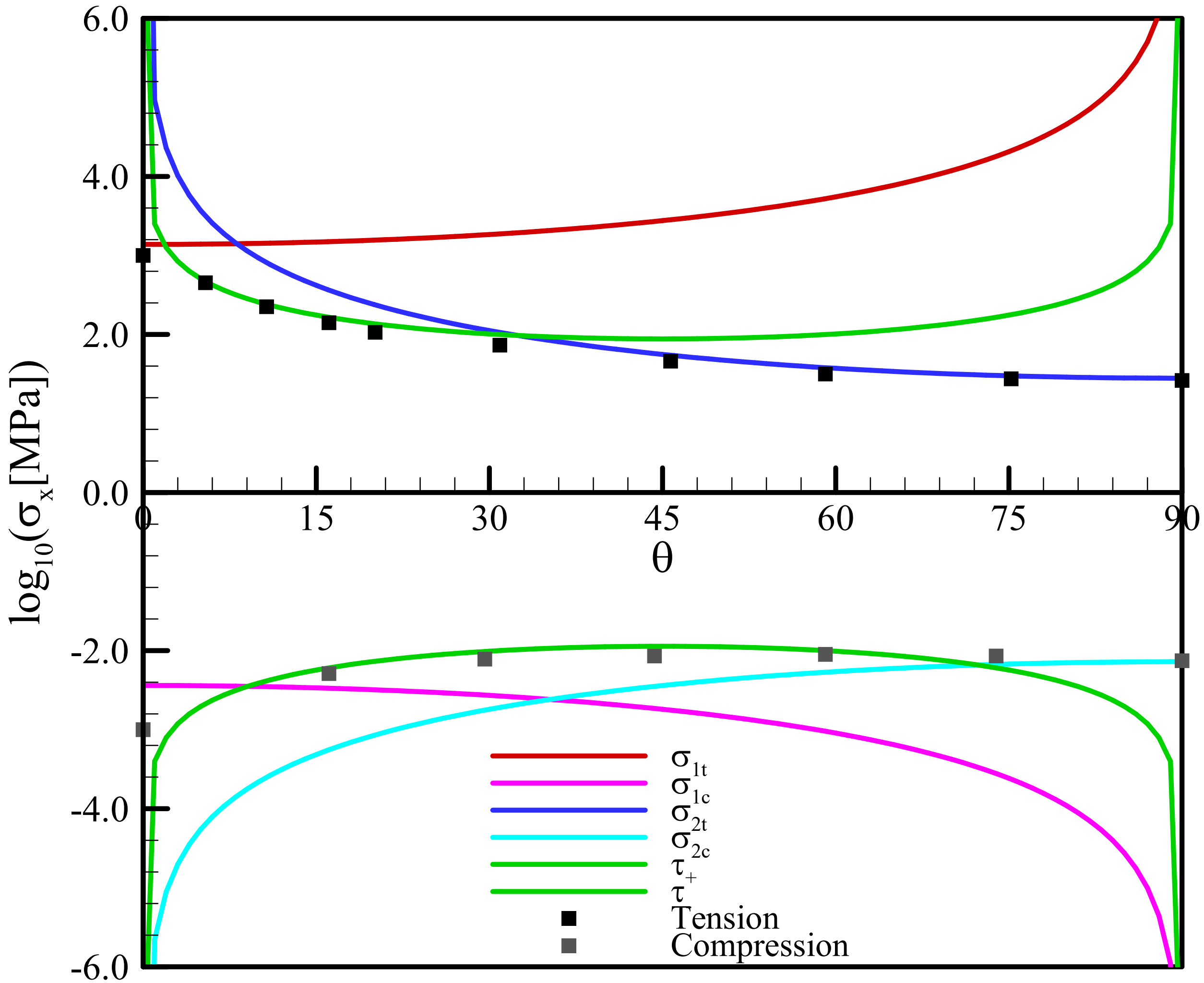
1) Maximum stress failure criterion



$$\sigma_1 = \sigma_x \cos^2\theta$$

$$\sigma_2 = \sigma_x \sin^2\theta$$

$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$



2) Maximum strain failure criterion

Tensile stresses: $\varepsilon_1 < X_{\varepsilon t}$, $\varepsilon_2 < Y_{\varepsilon t}$

Compressive stresses: $\varepsilon_1 > X_{\varepsilon c}$, $\varepsilon_2 > Y_{\varepsilon c}$

Shear: $|\gamma_{12}| < S_\varepsilon$

Assumption: linear elastic behavior up to failure

$$X_{\varepsilon t} = \frac{X_t}{E_1} \quad X_{\varepsilon c} = \frac{X_c}{E_1} \quad Y_{\varepsilon t} = \frac{Y_t}{E_2} \quad Y_{\varepsilon c} = \frac{Y_c}{E_2} \quad S_\varepsilon = \frac{S}{G_{12}}$$

2) Maximum strain failure criterion

$$\sigma_1 = \sigma_x \cos^2\theta$$
$$\sigma_2 = \sigma_x \sin^2\theta$$
$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$

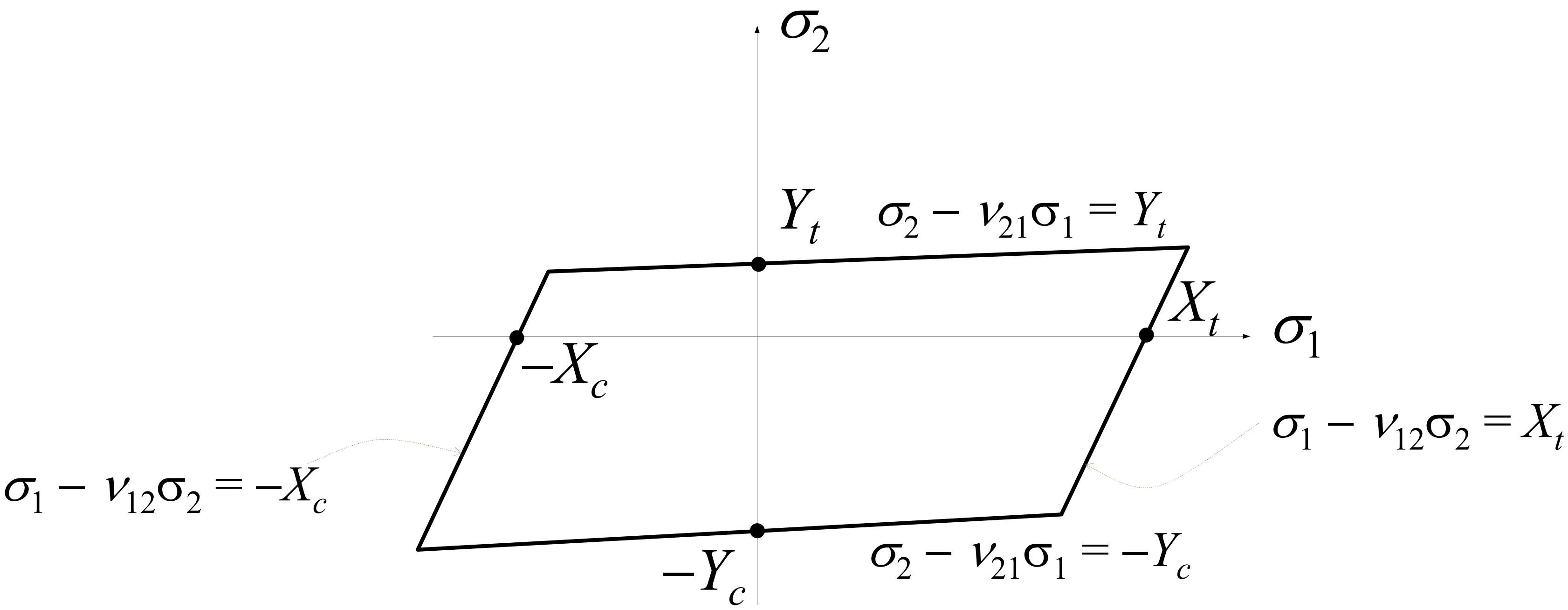
$$\varepsilon_1 = (\sigma_1 - \nu_{12}\sigma_2)/E_1$$
$$\varepsilon_2 = (\sigma_2 - \nu_{21}\sigma_1)/E_2$$
$$\gamma_{12} = \tau_{12}/G_{12}$$

$$X_{\varepsilon t} = \frac{X_t}{E_1} \quad X_{\varepsilon c} = \frac{X_c}{E_1}$$
$$Y_{\varepsilon t} = \frac{Y_t}{E_2} \quad Y_{\varepsilon c} = \frac{Y_c}{E_2} \quad S_\varepsilon = \frac{S}{G_{12}}$$

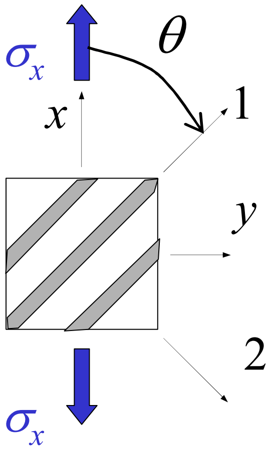
$$\frac{X_c}{(\cos^2\theta - \nu_{12}\sin^2\theta)} < \sigma_x < \frac{X_t}{(\cos^2\theta - \nu_{12}\sin^2\theta)}$$
$$\frac{Y_c}{(\sin^2\theta - \nu_{21}\cos^2\theta)} < \sigma_x < \frac{Y_t}{(\sin^2\theta - \nu_{21}\cos^2\theta)}$$
$$|\sigma_x| < \left|\frac{S}{\sin\theta\cos\theta}\right|$$

2) Maximum strain failure criterion

Failure envelope



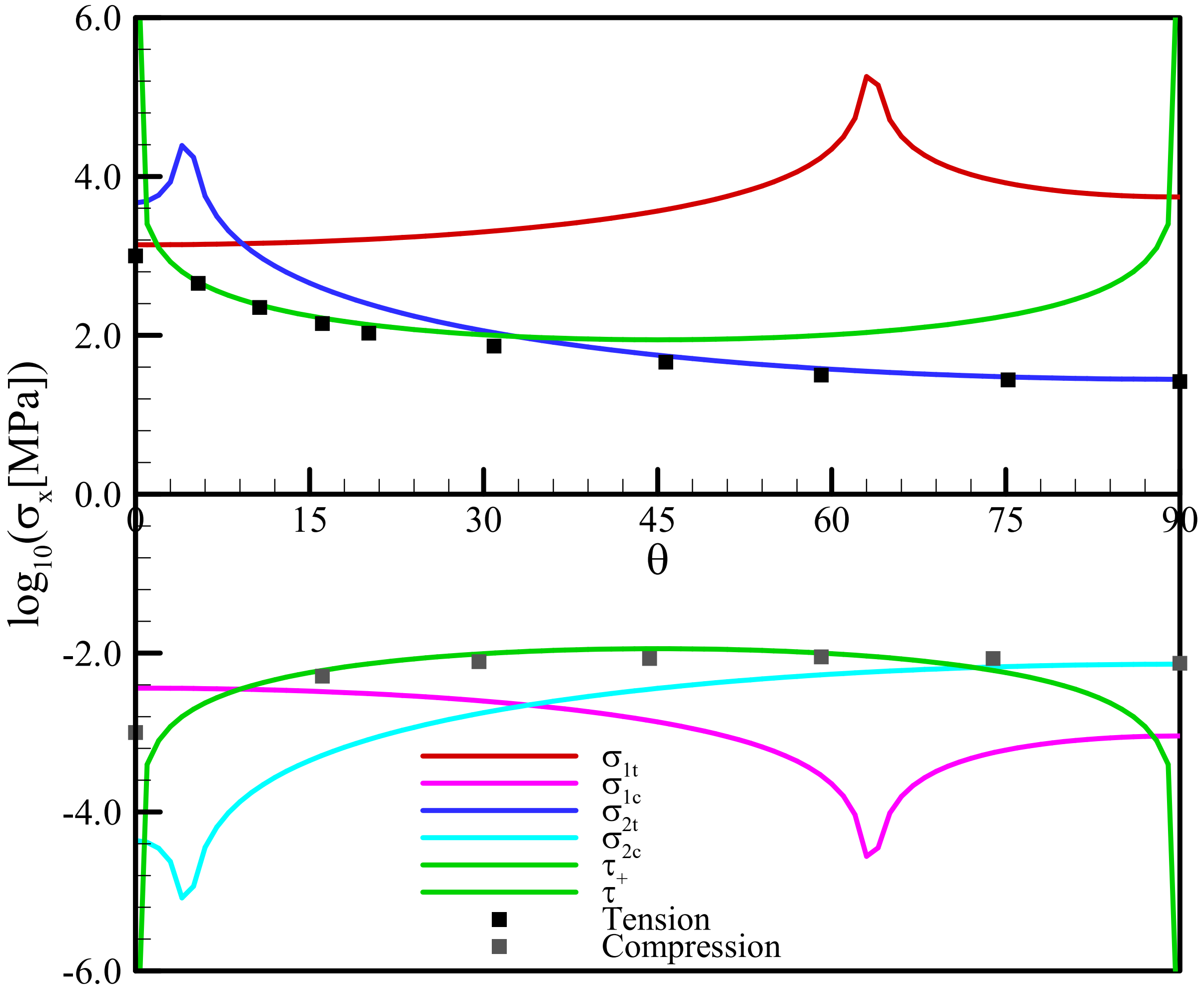
2) Maximum strain failure criterion



$$\sigma_1 = \sigma_x \cos^2 \theta$$

$$\sigma_2 = \sigma_x \sin^2 \theta$$

$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$



3) Tsai-Hill failure criterion

Extension of von Mises yield criterion for metals

Von Mises criterion can be related to the amount of energy required to distort an isotropic body rather than to dilate it

However, distortion and dilatation cannot be separated in orthotropic materials so the Tsai-Hill criterion cannot be strictly called a distortional energy failure criterion

The constants *F*, *G*, *H*, *L*, *M* and *N* are interpreted as strengths that must be experimentally obtained

$$(G+H)\sigma_1^2+(F+H)\sigma_2^2+(F+G)\sigma_3^2-2H\sigma_1\sigma_2-2G\sigma_1\sigma_3-2F\sigma_2\sigma_3+$$
$$2L\tau_{23}^2+2M\tau_{13}^2+2N\tau_{12}^2=1$$

3) Tsai-Hill failure criterion

If only τ_{12} acts on the body then $2N = 1/S^2$

If only σ_1 acts on the body then $G + H = 1/X^2$

If only σ_2 acts on the body then $F + H = 1/Y^2$

If only σ_3 acts on the body then $F + G = 1/Z^2$

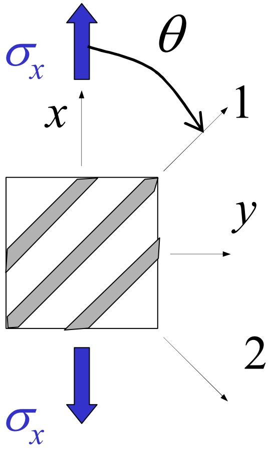
$$2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \qquad 2G = \frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2} \qquad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}$$

If 23 is a plane of transverse isotropy then $Z = Y$

In a plane stress state $\sigma_3 = \tau_{23} = \tau_{13} = 0$

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1 \sigma_2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\tau_{12}^2}{S^2} = 1$$

3) Tsai-Hill failure criterion



$$\sigma_1 = \sigma_x \cos^2\theta$$
$$\sigma_2 = \sigma_x \sin^2\theta$$
$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1\sigma_2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\tau_{12}^2}{S^2} = 1$$

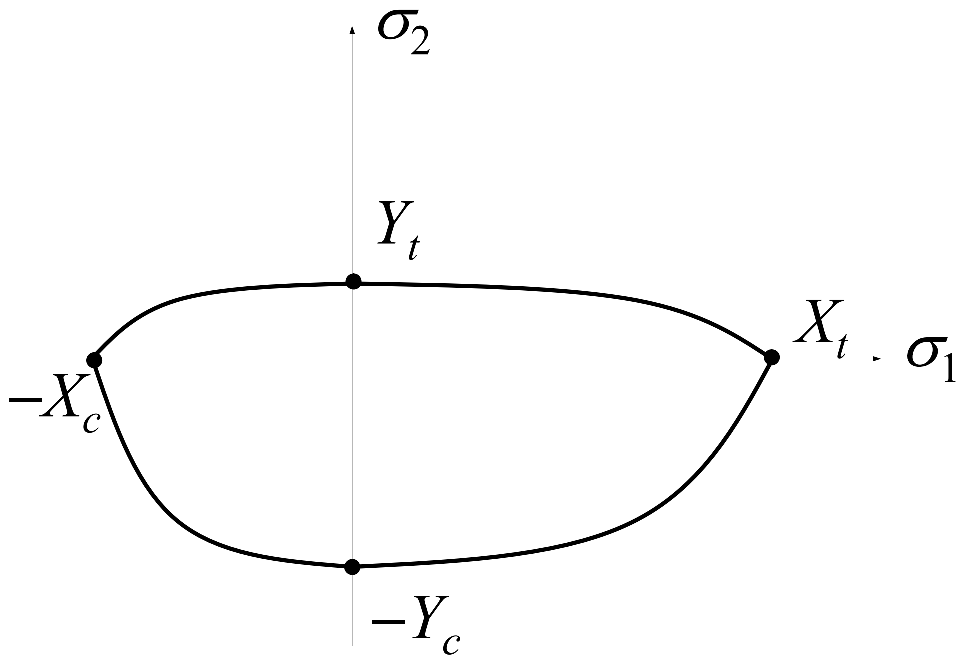
$$\frac{1}{\sigma_x^2} = \frac{\cos^4\theta}{X^2} + \left(\frac{1}{S^2} - \frac{1}{X^2}\right)\cos^2\theta\sin^2\theta + \frac{\sin^4\theta}{Y^2}$$

X and Y assume the appropriate values depending on the sign of σ_1 and σ_2

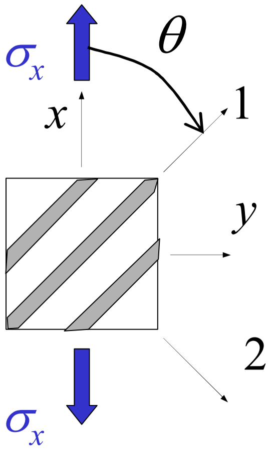
The failure envelope in stress space consists of four curves connected but that are not continuously differentiable

3) Tsai-Hill failure criterion

Envelope composed by four arcs of ellipses



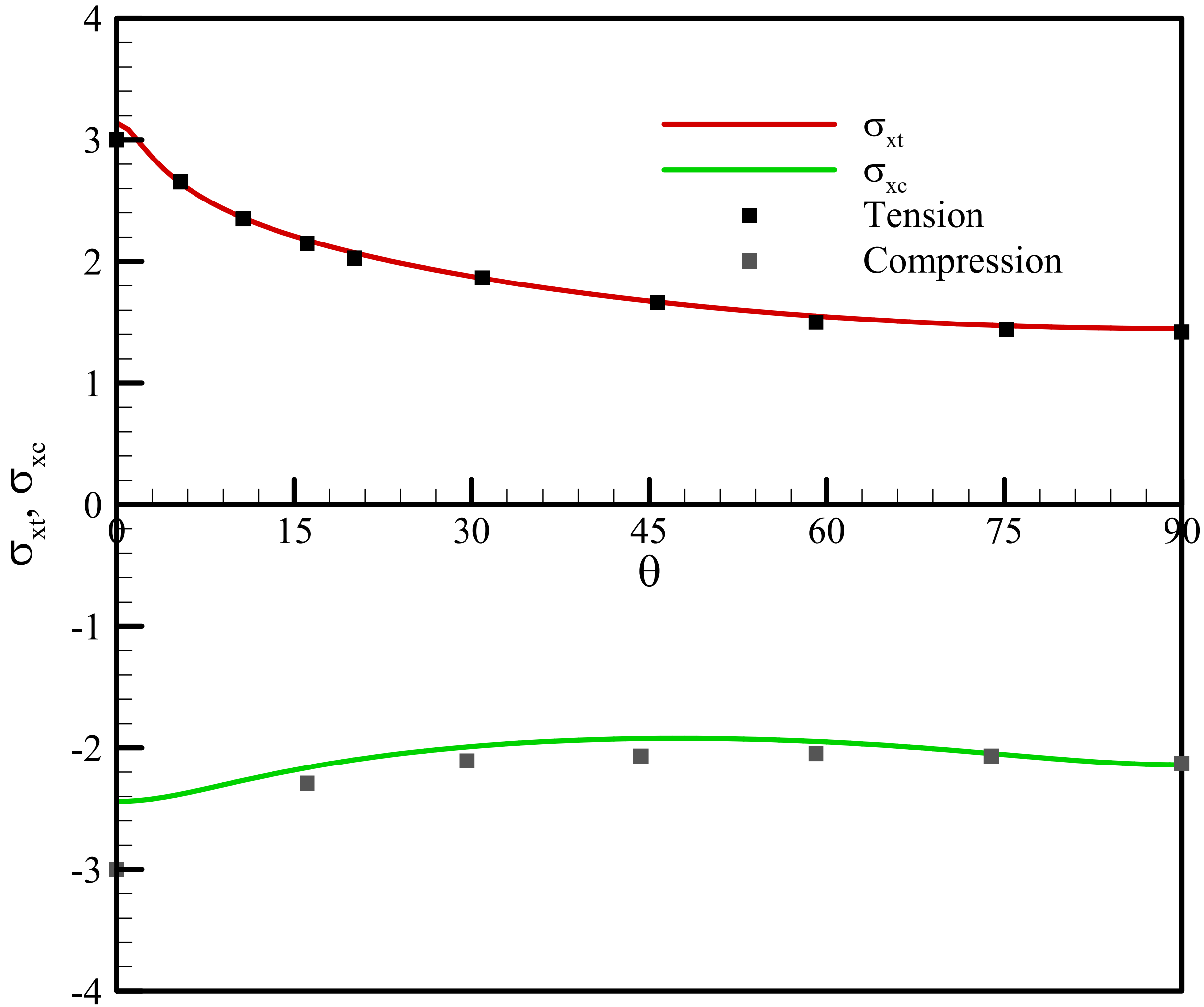
3) Tsai-Hill failure criterion



$$\sigma_1 = \sigma_x \cos^2 \theta$$

$$\sigma_2 = \sigma_x \sin^2 \theta$$

$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$



4) Hoffman failure criterion

Addition of linear terms to Tsai-Hill criterion to account for differences in tension and compression

9 strengths are required: X_t, X_c, Y_t, Y_c, Z_t, Z_c, S_{23}, S_{13}, S_{12}

$$C_1(\sigma_2 - \sigma_3)^2 + C_2(\sigma_3 - \sigma_1)^2 + C_3(\sigma_2 - \sigma_1)^2 + C_4\sigma_1 + C_5\sigma_2 + C_6\sigma_3 +$$
$$C_7\tau_{23}^2 + C_8\tau_{13}^2 + C_9\tau_{12}^2 = 1$$

Plane stress state $\sigma_3 = \tau_{23} = \tau_{13} = 0$

Transverse isotropy in the 23 plane: $Z_t = Y_t$, $Z_c = Y_c$, $S_{13} = S_{12}$

$$\frac{\sigma_1^2}{X_c X_t} - \frac{\sigma_1\sigma_2}{X_c X_t} + \frac{\sigma_2^2}{Y_c Y_t} - \frac{X_t - X_c}{X_c X_t}\sigma_1 - \frac{Y_t - Y_c}{Y_c Y_t}\sigma_2 + \frac{\tau_{12}^2}{S^2} = 1$$

5) Tsai-Wu tensor failure criterion

Full quadratic polynomial

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \qquad i, j = 1,\ldots,6$$

$\sigma_4 = \tau_{23},\ \sigma_5 = \tau_{13},\ \sigma_6 = \tau_{12}$

Orthotropic lamina under plane stress:

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + 2F_{12}\sigma_1\sigma_2 = 1$$

The terms F_6, F_{16} and F_{26} must be zero since failure is independent of the sign of the shear stress

5) Tsai-Wu tensor failure criterion

Pure tension along 1: $F_1 X_t + F_{11} X_t^2 = 1$

Pure compression along 1: $-F_1 X_c + F_{11} X_c^2 = 1$

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c} \qquad F_{11} = \frac{1}{X_t X_c}$$

Pure tension along 2: $F_2 Y_t + F_{22} Y_t^2 = 1$

Pure compression along 2: $-F_2 Y_c + F_{22} Y_c^2 = 1$

$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \qquad F_{22} = \frac{1}{Y_t Y_c}$$

Pure shear: $F_{66} = 1/S^2$

How to determine F_{12}? Uni-axial tests are not enough.

To guarantee that the conic curve is an ellipse $F_{11}F_{22} - (F_{12})^2 > 0$

Chose arbitrarily $F_{12} = 0$ or else $F_{12} = -(F_{11}F_{22})^{1/2}/2$

5) Tsai-Wu failure criterion

Envelope is an ellipse