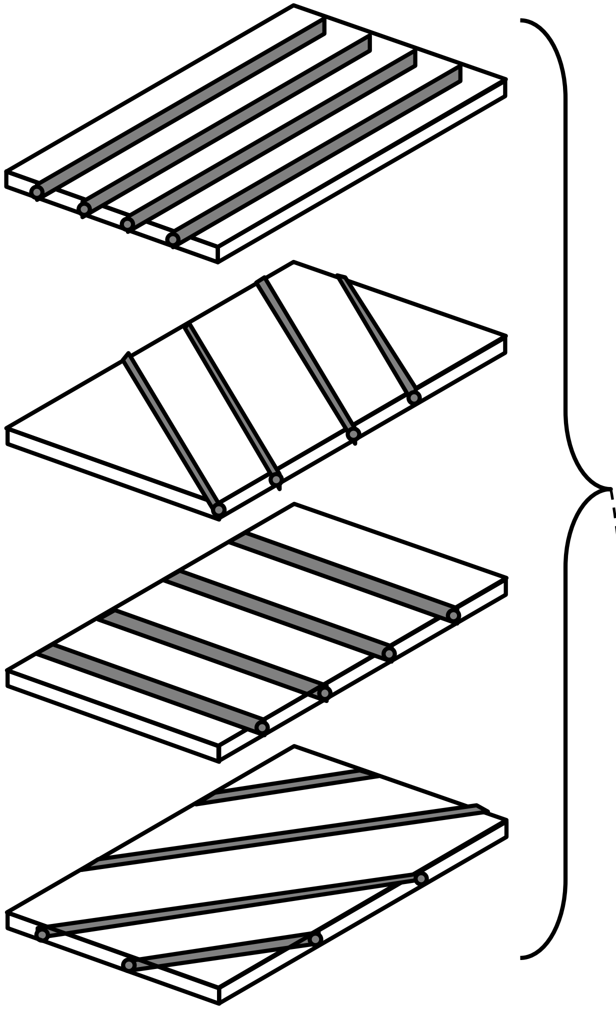
Analysis and design of composite structures

Class notes

4. Laminates

Introduction

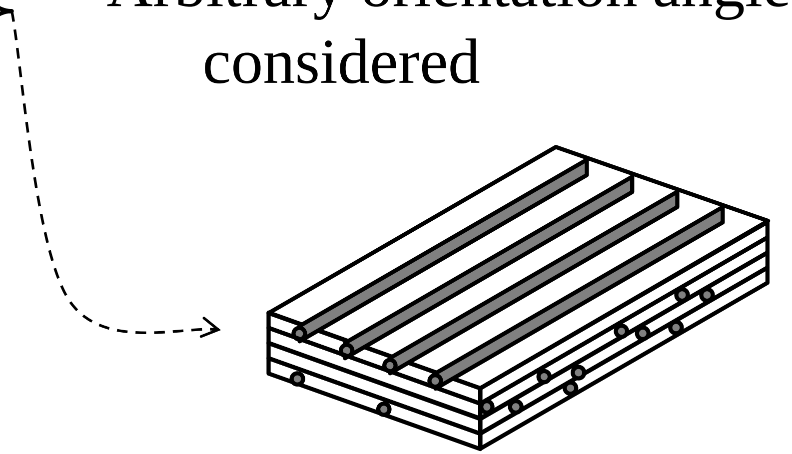


Two or more laminae bonded together

How will the laminate respond to loads?

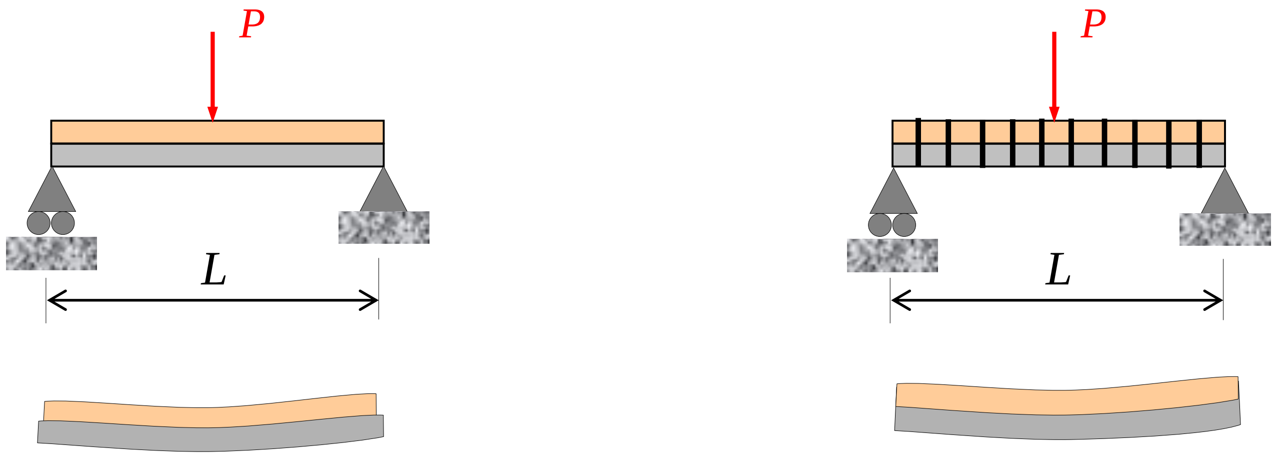
Laminate properties are obtained from the laminae properties

Arbitrary orientation angles must be considered



Introduction

Laminates have high bending stiffness



$$\Delta_1 = \frac{PL^3}{48EI} = \frac{PL^3}{2\times 48E\dfrac{bh^3}{12}} = \frac{PL^3}{8Ebh^3}$$

$$\Delta_2 = \frac{PL^3}{48EI'} = \frac{PL^3}{48E\dfrac{b(2h)^3}{12}} = \frac{PL^3}{32Ebh^3}$$

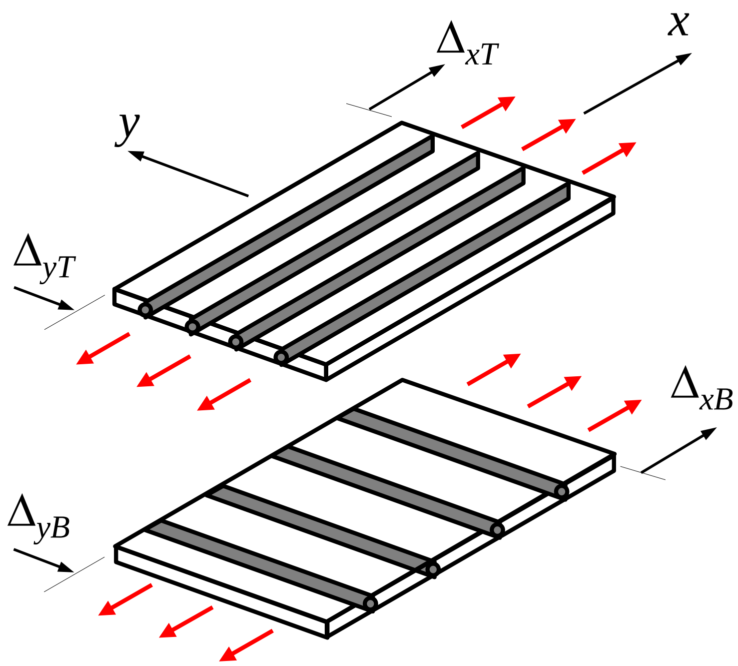
$$\Delta_1 = 4\Delta_2$$

Introduction

$$\sigma_{xT} = \frac{E_1}{E_2}\sigma_{xB}$$

No bonding $\Rightarrow \Delta_{yT} = \frac{E_1}{E_2}\Delta_{yB}$

Bonding $\Rightarrow \sigma_{yT} = -\sigma_{yB}$



Conditions to be satisfied:

- deformation compatibility
- stress × strain relations
- equilibrium

4.1 Bending of thin isotropic plates

Thin plates

Small thickness compared to other dimensions

Resists bending and membrane loads

Aeronautical panels

Different loadings and boundary conditions

Equilibrium described by fourth order differential equation

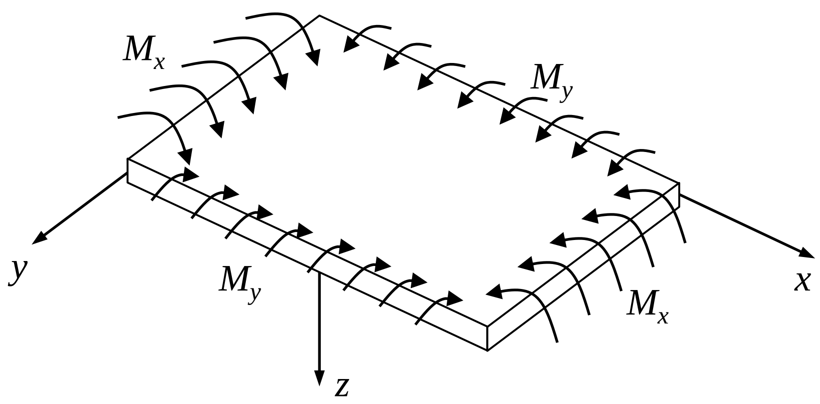
Pure bending of thin plates

Bending moments M_x and M_y

Positive bending when they compress bottom surface

Plane sections remain plane after deformation

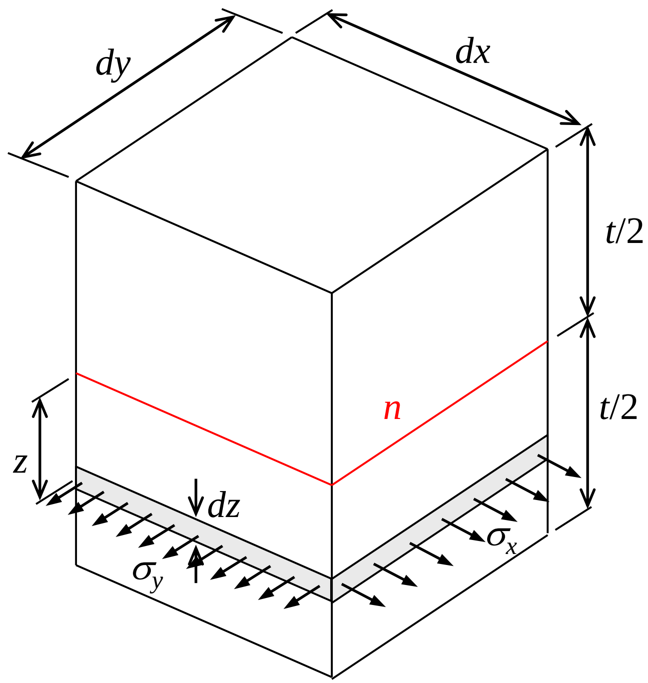
The mid plane does not deform

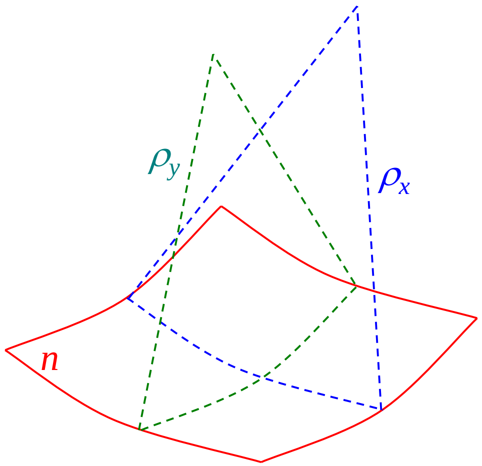


Pure bending of thin plates

Curvature radii ρ_x and ρ_y on plane *xz* and *yz*

Positive bending moments ⇒ positive curvatures

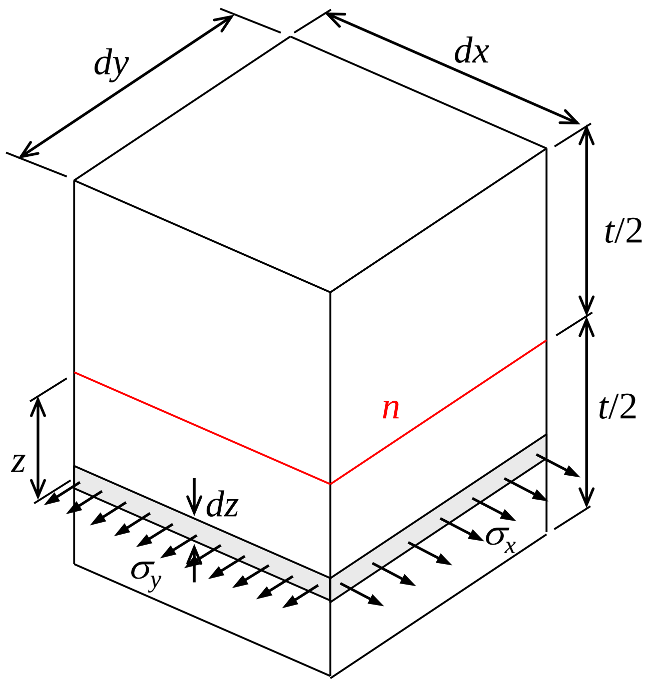


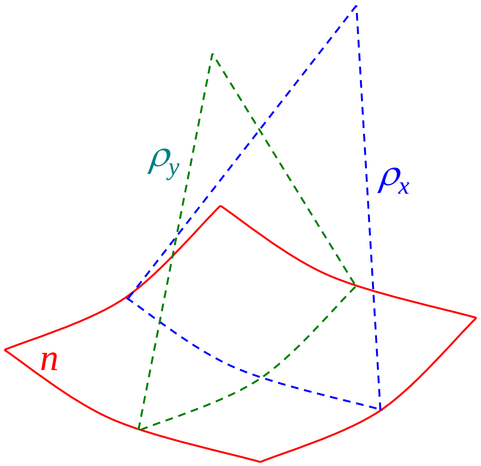


Pure bending of thin plates

Stresses and strains

$$\varepsilon_x = \frac{z}{\rho_x} \quad , \quad \varepsilon_y = \frac{z}{\rho_y} \quad \Rightarrow \quad \sigma_x = \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right) \quad , \quad \sigma_y = \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right)$$





Pure bending of thin plates

Moment and curvature: isotropic plate

$$M_x = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2}\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)dz = D\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)$$

$$M_y = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2}\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right)dz = D\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right)$$

$$D = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2}dz = \frac{Et^3}{12(1-\nu^2)}$$

plate bending stiffness

$$\frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2}, \qquad \frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

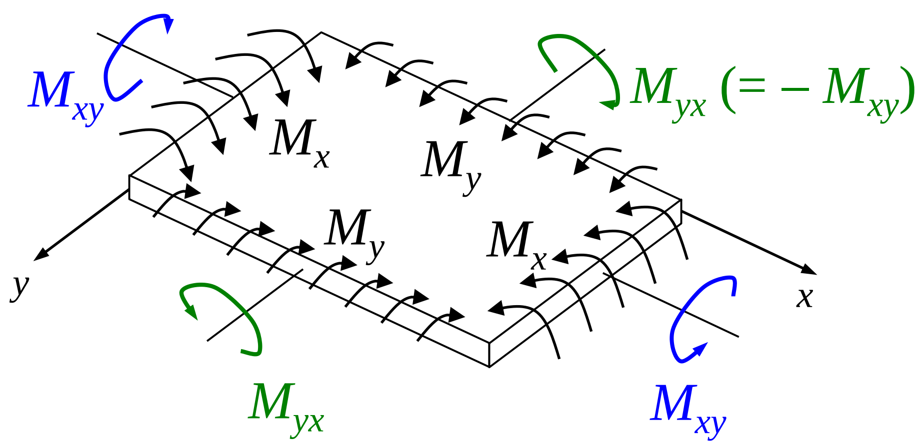
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$$

Plate subject to bending and torsion

In the most general case there are tangential components

Tangent components produce torsion

M_{xy} and M_{yx} are consequence of $\tau_{xy} \Rightarrow M_{xy} = -M_{yx}$



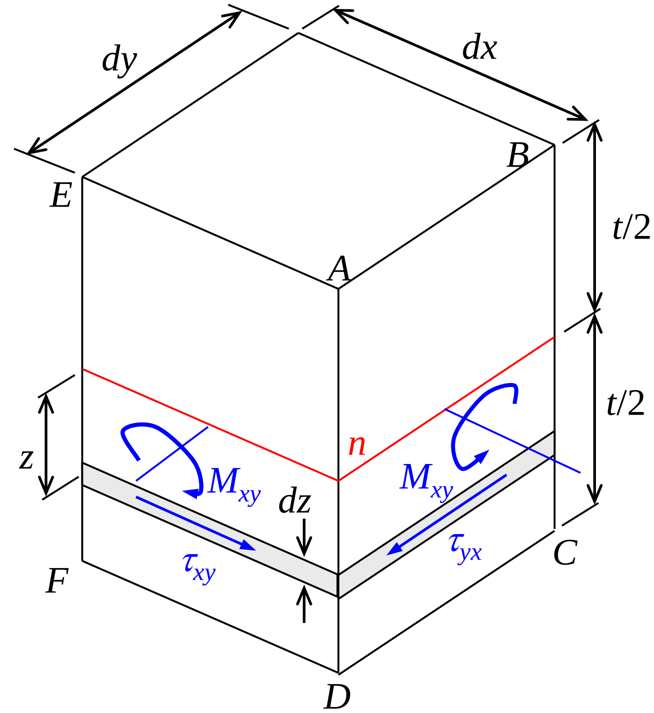


Plate subject to bending and torsion

Moments M_x, M_y and M_{xy}

Decomposition in M_t and M_n

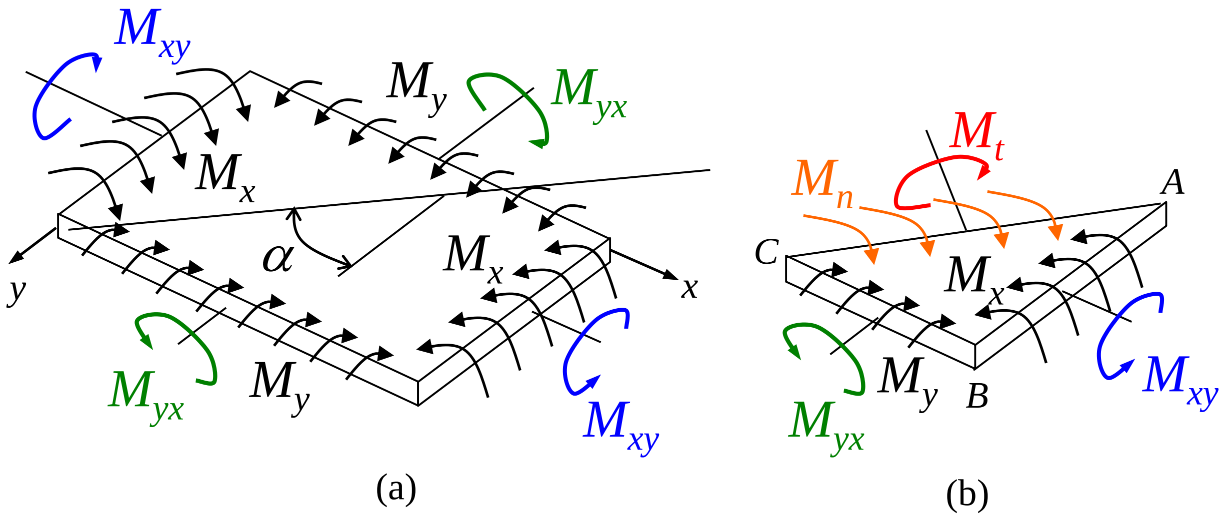
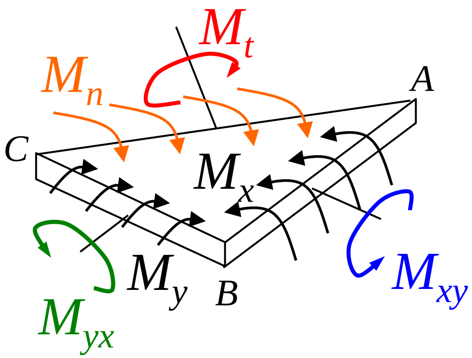


Plate subject to bending and torsion

Moment transformation



$$M_n \overline{AC} = M_x \overline{AB} \cos\alpha + M_y \overline{BC} \sin\alpha - M_{xy} \overline{AB} \sin\alpha - M_{xy} \overline{BC} \cos\alpha$$

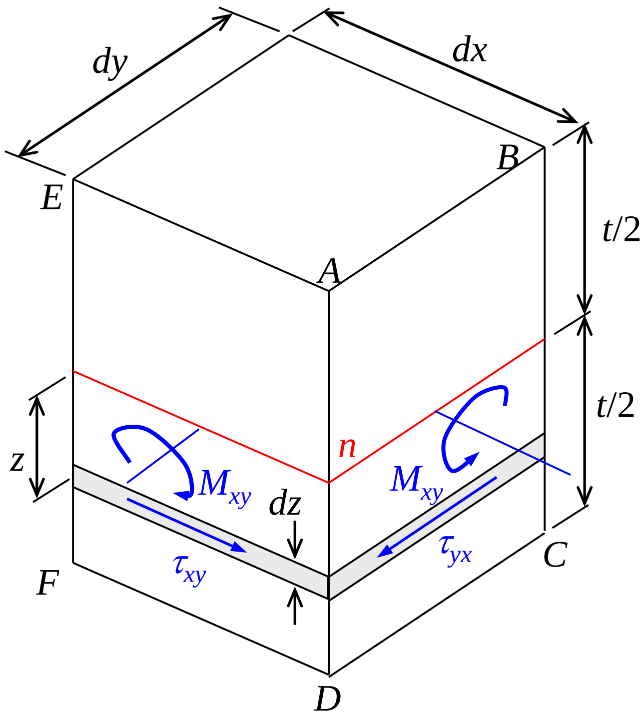
$$M_n = M_x \cos^2\alpha + M_y \sin^2\alpha - M_{xy} \sin 2\alpha$$

$$M_t \overline{AC} = M_x \overline{AB} \sin\alpha - M_y \overline{BC} \cos\alpha + M_{xy} \overline{AB} \cos\alpha - M_{xy} \overline{BC} \sin\alpha$$

$$M_t = \frac{M_x - M_y}{2} \sin 2\alpha + M_{xy} \cos 2\alpha$$

Plate subject to bending and torsion

Torsion moment M_{xy}



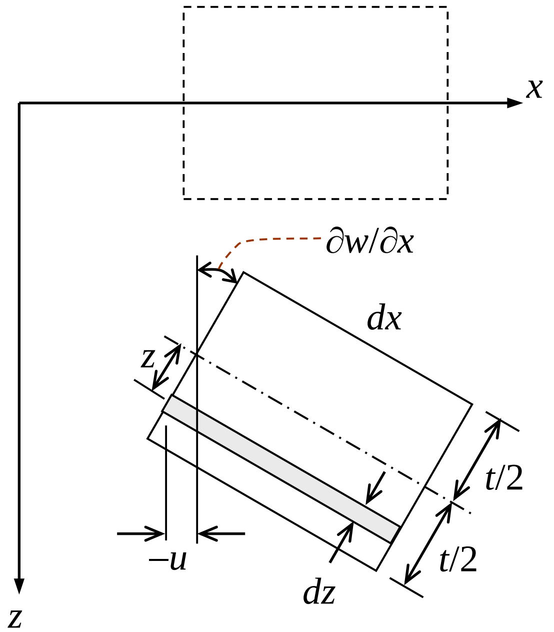
$$M_{xy}dy = -\int_{-t/2}^{t/2} \tau_{xy} z dy dz \quad \Rightarrow \quad M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} z dz$$

$$M_{xy} = -G\int_{-t/2}^{t/2} \gamma_{xy} z dz$$

How to relate M_{xy} and w?

Plate subject to bending and torsion

Determination of shear strain γ_{xy}



$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y} \quad \Rightarrow \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = -G\int_{-t/2}^{t/2} \gamma_{xy} z dz = G\int_{-t/2}^{t/2} 2z^2 \frac{\partial^2 w}{\partial x \partial y} dz = \frac{Gt^3}{6}\frac{\partial^2 w}{\partial x \partial y} = \frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

4.2. Classical laminate theory

Lamina stress × strain behavior

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

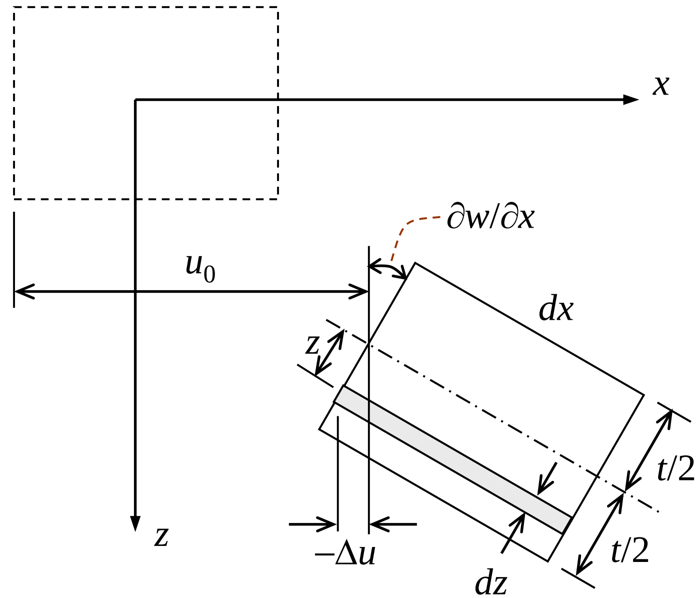
→ For layer k: $\{\sigma\}_k = [\overline{Q}]_k \{\varepsilon\}_k$

Stress and strain variation in the laminate

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w}{\partial x}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w}{\partial y}$$

$$w(x,y,z) = w(x,y)$$



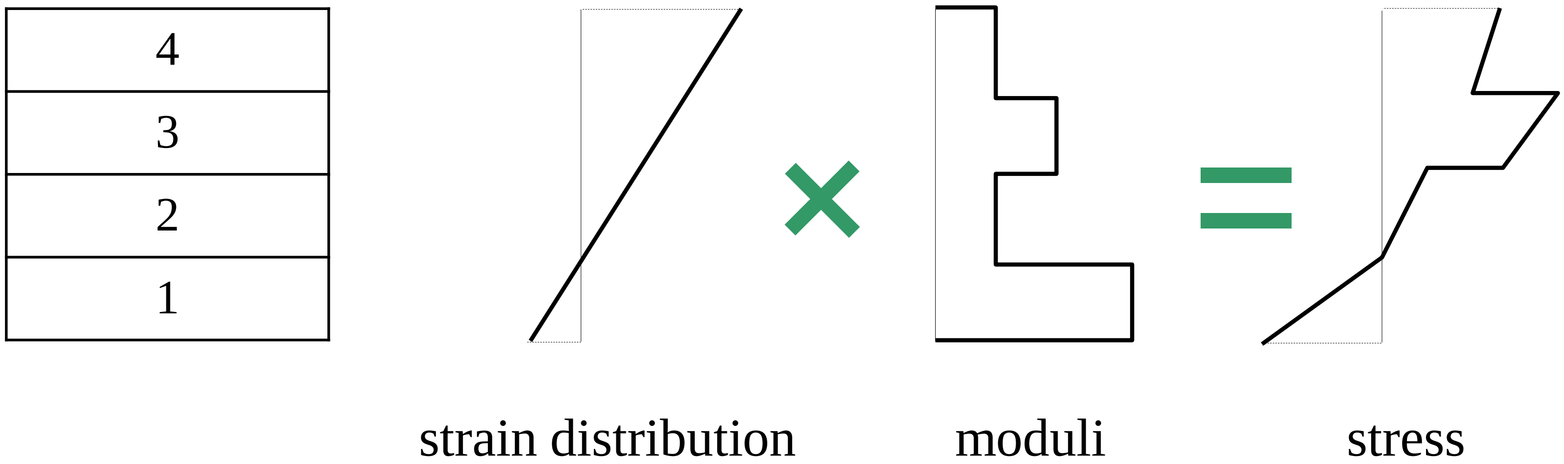
$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x^0 + z\kappa_x$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_y^0 + z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \gamma_{xy}^0 + z\kappa_{xy}$$

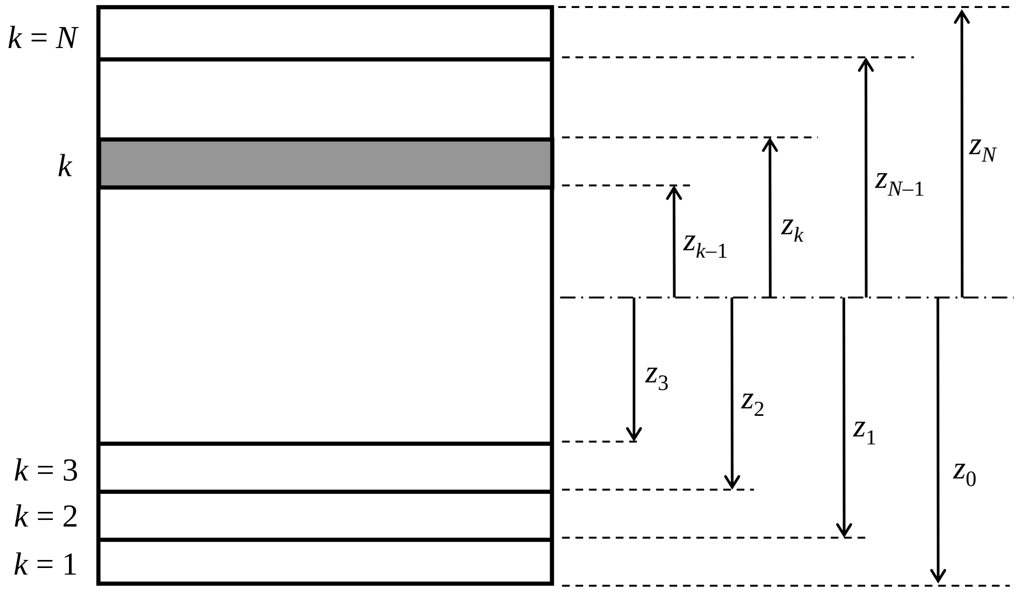
$$\Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Stress and strain variation



Resultant laminate forces and moments

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz \qquad \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\,dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k z\,dz$$



Resultant laminate forces and moments

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) zdz$$

Laminate matrices

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k dz = \sum_{k=1}^{N} (z_k - z_{k-1}) \left[\overline{Q}\right]_k$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k z\,dz = \frac{1}{2} \sum_{k=1}^{N} (z_k^2 - z_{k-1}^2) \left[\overline{Q}\right]_k$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k z^2 dz = \frac{1}{3} \sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \left[\overline{Q}\right]_k$$

Resultant laminate forces and moments

shear-extension coupling

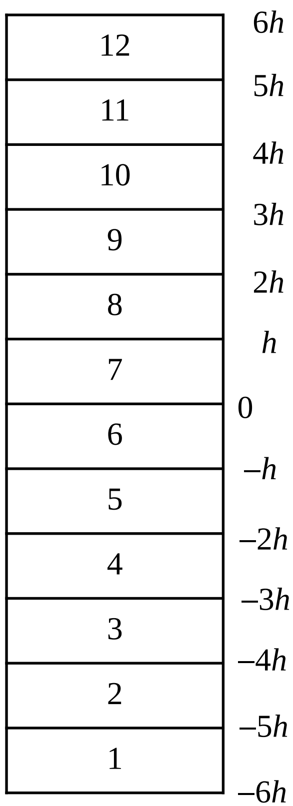
membrane-bending coupling

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

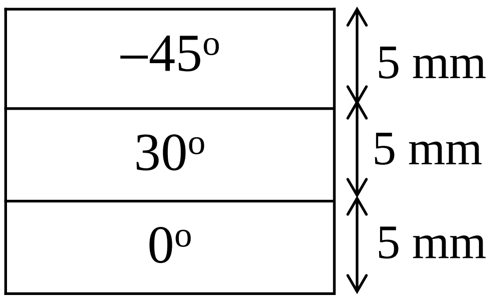
bend-twist coupling

Laminate matrices: weights



k	z_{k-1}	z_k	$(z_k - z_{k-1})$	$[z_k^2 - (z_{k-1})^2]/2$	$[z_k^3 - (z_{k-1})^3]/3$
1	$-6h$	$-5h$	h	$-5.5h^2$	$91h^3/3$
2	$-5h$	$-4h$	h	$-4.5h^2$	$61h^3/3$
3	$-4h$	$-3h$	h	$-3.5h^2$	$37h^3/3$
4	$-3h$	$-2h$	h	$-2.5h^2$	$19h^3/3$
5	$-2h$	$-h$	h	$-1.5h^2$	$7h^3/3$
6	$-h$	0	h	$-0.5h^2$	$h^3/3$
7	0	h	h	$0.5h^2$	$h^3/3$
8	h	$2h$	h	$1.5h^2$	$7h^3/3$
9	$2h$	$3h$	h	$2.5h^2$	$19h^3/3$
10	$3h$	$4h$	h	$3.5h^2$	$37h^3/3$
11	$4h$	$5h$	h	$4.5h^2$	$61h^3/3$
12	$5h$	$6h$	h	$5.5h^2$	$91h^3/3$

Example: Find the laminate stiffness matrices [*A*], [*B*] and [*D*] for a three-ply [0/30/–45] graphite/epoxy laminate. The graphite/epoxy mechanical properties are: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28 and G_{12} = 7.17 GPa.



From the constitutive relations for an orthotropic lamina under plane stress, we have:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

The stiffness matrix can be written as a function of the stiffness in the lamina principal directions as:

$$[\overline{Q}]_k = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}$$

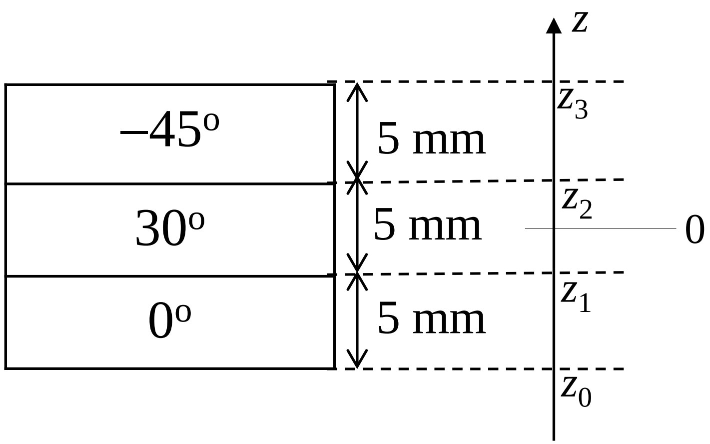
where:

$c = \cos\theta$; $s = \sin\theta$

Therefore, the coefficients of $[Q]_k$ are:

$$\begin{cases} Q_{xx} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 \\ Q_{yy} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 \\ Q_{xy} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4) \\ Q_{ss} = (Q_{11} + Q_{22} - 2Q_{12})c^2s^2 + Q_{66}(c^2 - s^2)^2 \\ Q_{xs} = (Q_{11} - Q_{12})c^3s + (Q_{12} - Q_{22})cs^3 - 2Q_{66}cs(c^2 - s^2) \\ Q_{ys} = (Q_{11} - Q_{12})cs^3 + (Q_{12} - Q_{22})c^3s + 2Q_{66}cs(c^2 - s^2) \end{cases}$$

Example: Find the laminate stiffness matrices [*A*], [*B*] and [*D*] for a three-ply [0/30/–45] graphite/epoxy laminate. The graphite/epoxy mechanical properties are: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28 and G_{12} = 7.17 GPa.



The positions of the top laminae faces are:

z_0 = –0.0075 m, z_1 = –0.0025 m

z_2 = 0.0025 m, z_3 = 0.0075 m

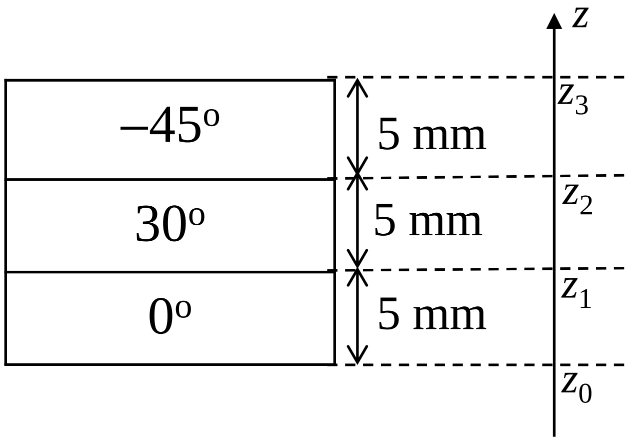
Replacing the angles of the laminae (0°, 30° and –45°) yields:

$$[\bar{Q}]_{-45} = \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \times 10^9$$

$$[\bar{Q}]_{30} = \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \times 10^9$$

$$[\bar{Q}]_{-45} = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \times 10^9$$

Example: Find the laminate stiffness matrices [*A*], [*B*] and [*D*] for a three-ply [0/30/–45] graphite/epoxy laminate. The graphite/epoxy mechanical properties are: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28 and G_{12} = 7.17 GPa.



The positions of the top laminae faces are:

z_0 = –0.0075 m, z_1 = –0.0025 m

z_2 = 0.0025 m, z_3 = 0.0075 m

Integrating through the thickness yields:

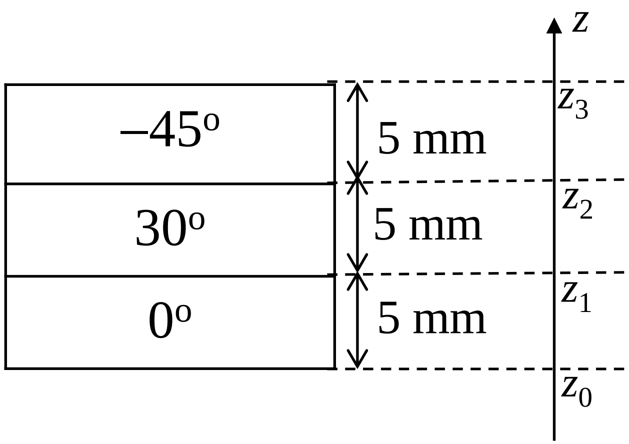
$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} dz =$$

$$\sum_{K=1}^{n} (z_k - z_{k-1})[\overline{Q}]_k = \begin{bmatrix} 17.39 & 3.88 & 0.566 \\ 3.88 & 4.533 & -1.141 \\ 0.566 & -1.141 & 4.525 \end{bmatrix} \times 10^8$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} z dz =$$

$$\frac{1}{2} \sum_{K=1}^{n} (z_k^2 - z_{k-1}^2)[\overline{Q}]_k = \begin{bmatrix} -3.129 & 0.985 & -1.072 \\ 0.985 & 1.158 & -1.072 \\ -1.072 & -1.072 & 0.985 \end{bmatrix} \times 10^6$$

Example: Find the laminate stiffness matrices [*A*], [*B*] and [*D*] for a three-ply [0/30/–45] graphite/epoxy laminate. The graphite/epoxy mechanical properties are: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28 and G_{12} = 7.17 GPa.



The positions of the top laminae faces are:

z_0 = –0.0075 m, z_1 = –0.0025 m

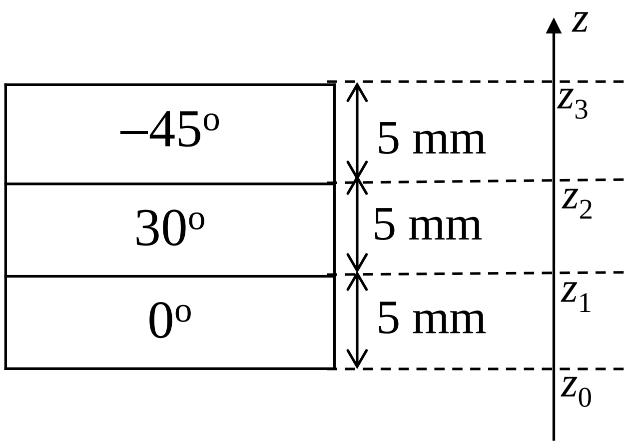
z_2 = 0.0025 m, z_3 = 0.0075 m

Integrating through the thickness yields:

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} z^2 dz =$$

$$\frac{1}{3}\sum_{K=1}^{n} (z_k^3 - z_{k-1}^3)[\overline{Q}]_k = \begin{bmatrix} 33.43 & 6.461 & -5.240 \\ 6.461 & 9.320 & -5.596 \\ -5.240 & -5.596 & 7.663 \end{bmatrix} \times 10^3$$

Example: Find the laminate stiffness matrices $[A]$, $[B]$ and $[D]$ for a three-ply [0/30/–45] graphite/epoxy laminate. The graphite/epoxy mechanical properties are: E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28 and G_{12} = 7.17 GPa.



The positions of the top laminae faces are:

z_0 = –0.0075 m, z_1 = –0.0025 m

z_2 = 0.0025 m, z_3 = 0.0075 m

Integrating through the thickness yields:

$$[A]=\begin{bmatrix} 17.39 & 3.88 & 0.566 \\ 3.88 & 4.533 & -1.141 \\ 0.566 & -1.141 & 4.525 \end{bmatrix}\times 10^{8}$$

shear-extension coupling

$$[B]=\begin{bmatrix} -3.129 & 0.985 & -1.072 \\ 0.985 & 1.158 & -1.072 \\ -1.072 & -1.072 & 0.985 \end{bmatrix}\times 10^{6}$$

membrane-bending coupling

bend-twist coupling

$$[D]=\begin{bmatrix} 33.43 & 6.461 & -5.240 \\ 6.461 & 9.320 & -5.596 \\ -5.240 & -5.596 & 7.663 \end{bmatrix}\times 10^{3}$$

4.3. Mindlin laminate plate theory

Assumptions

Domain: $\Omega = \{(x,y,z) \in \Re^3 | -t/2 \leq z \leq t/2,\ (x,y) \in \Re^2\}$

$\sigma_z = 0$

$\bar{u}(x,y,z) = u(x,y) + z\,\psi_x(x,y)$

$\bar{v}(x,y,z) = v(x,y) + z\,\psi_y(x,y)$

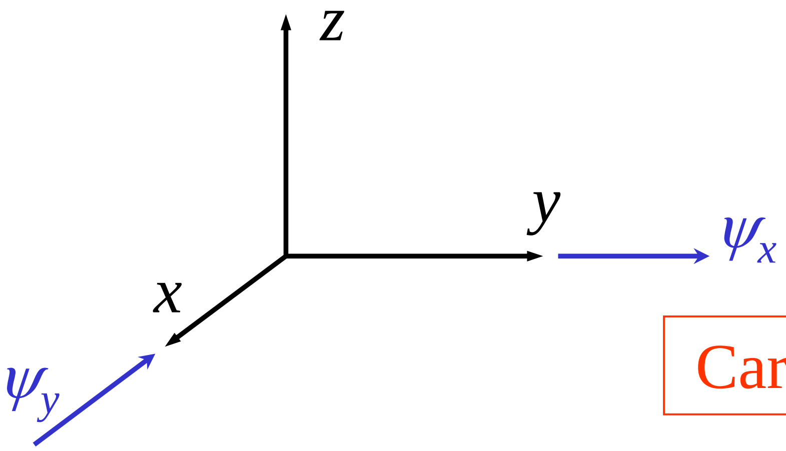
$\bar{w}(x,y,z) = w(x,y)$

Assumptions

The thickness t may be a function of x, y

$\sigma_z = 0$ is the plate stress assumption

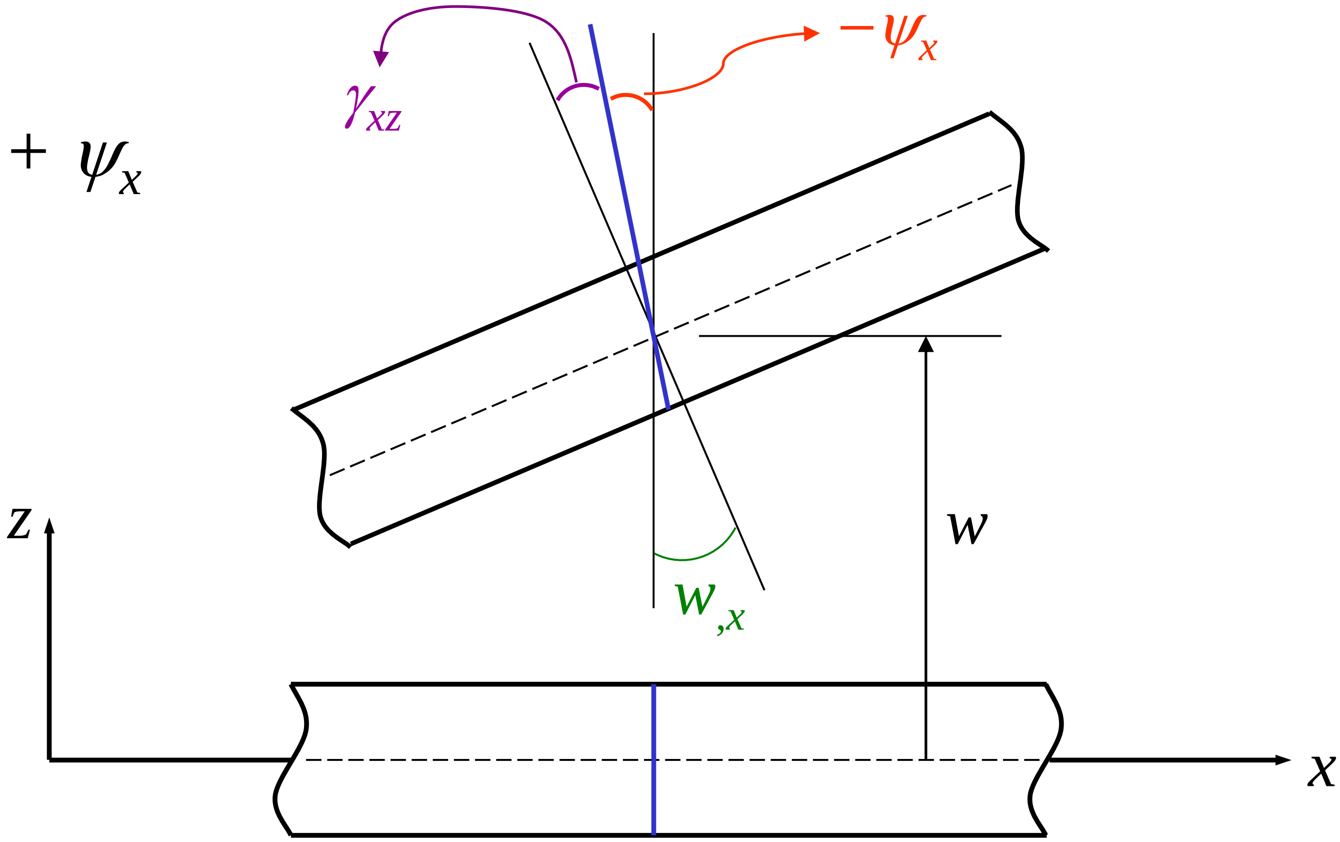
Plane sections remain plane but not normal to mid surface



Carefully check sign convention

Rotation of plane section

$$\gamma_{xz} = w_{,x} + \psi_x$$



Strain × displacement relations

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} = u_{,x} + z\psi_{x,x}$$

$$\varepsilon_{yy} = \frac{\partial \bar{v}}{\partial y} = v_{,y} + z\psi_{y,y}$$

$$\gamma_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = u_{,y} + v_{,x} + \frac{z}{2}(\psi_{x,y} + \psi_{y,x})$$

$$\varepsilon_{zz} = \frac{\partial \bar{w}}{\partial z} = w_{,z} = 0$$

$$\gamma_{yz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} = w_{,y} + \psi_{y}$$

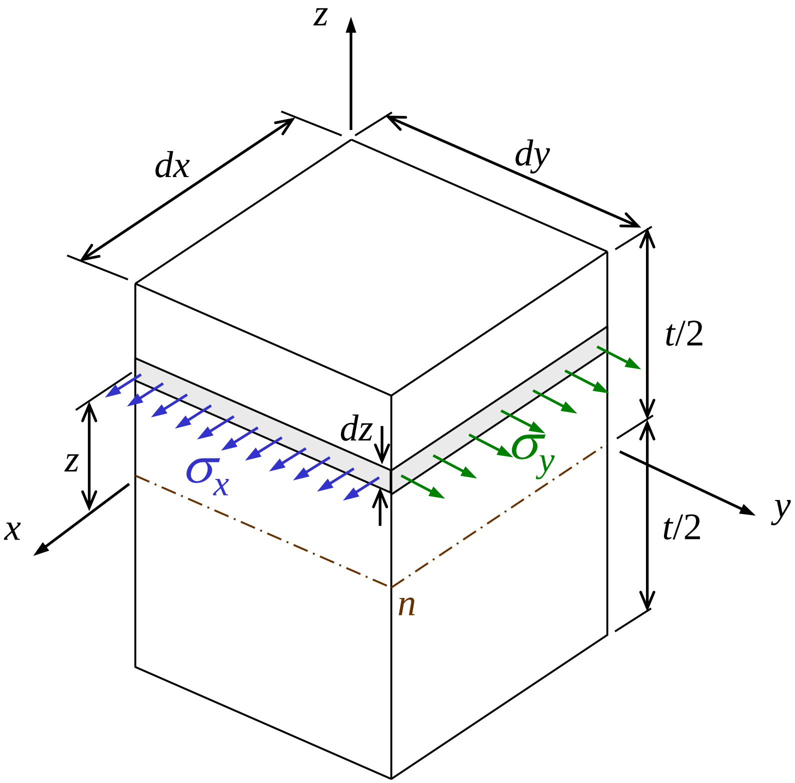
$$\gamma_{xz} = \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} = w_{,x} + \psi_{x}$$

u, v in-plane displacements

w transverse displacement

ψ_α rotation angle

Plate infinitesimal element: stress distributions



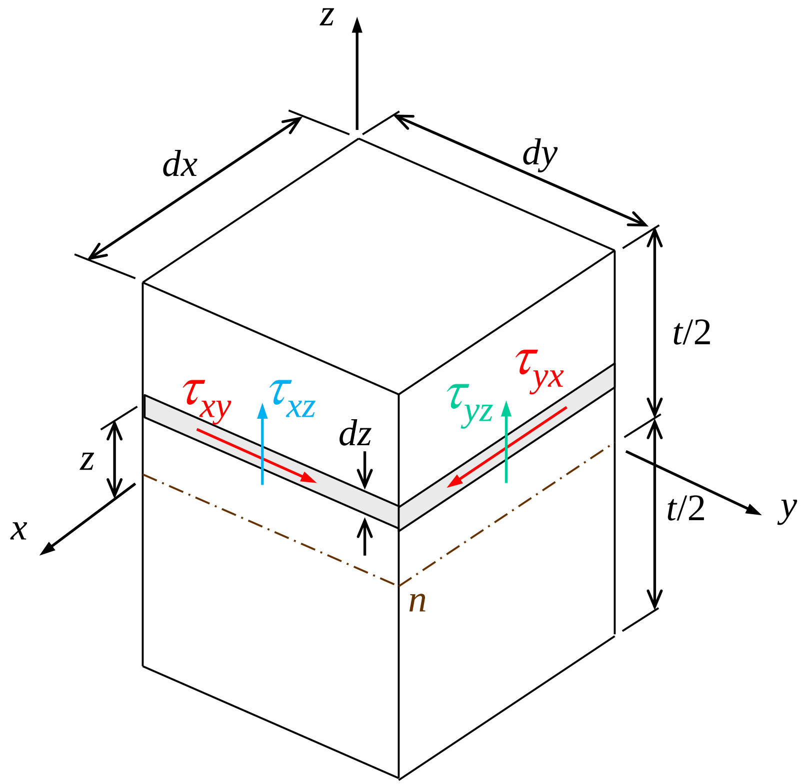


Plate resultant forces and moments

Membrane forces

$$N_x = \int_{-t/2}^{t/2} \sigma_x dz$$

$$N_y = \int_{-t/2}^{t/2} \sigma_y dz$$

$$N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz$$

Moments

$$M_x = \int_{-t/2}^{t/2} z\sigma_x dz$$

$$M_y = \int_{-t/2}^{t/2} z\sigma_y dz$$

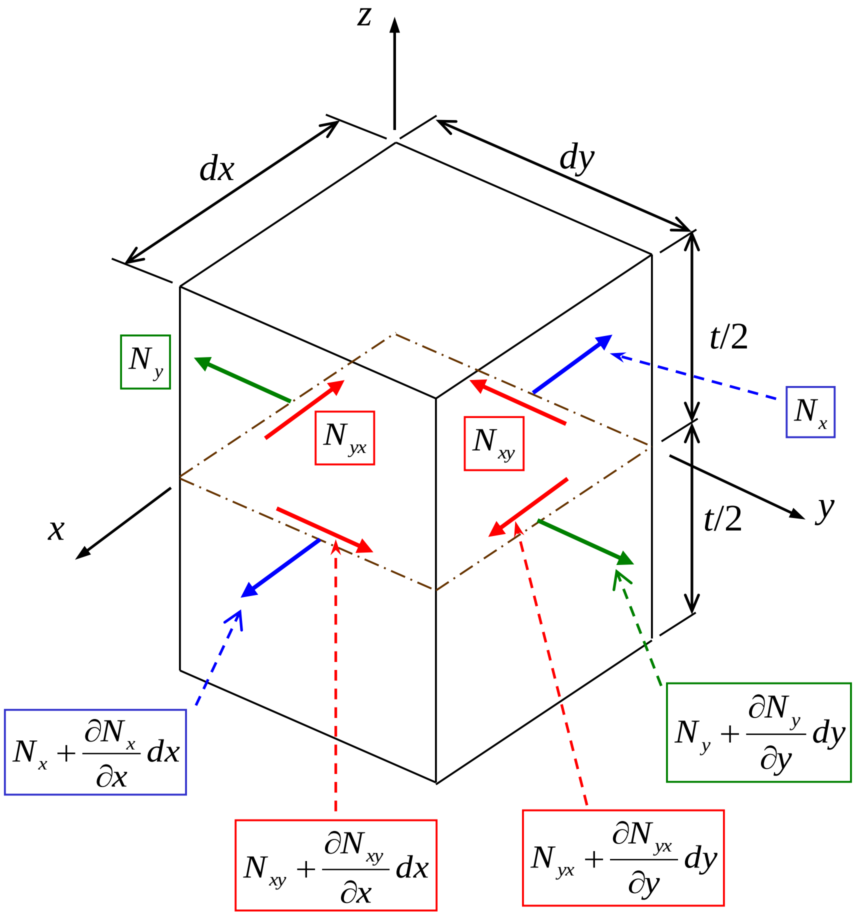
$$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz$$

Shear forces

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz$$

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz$$

Plate infinitesimal element: internal membrane forces

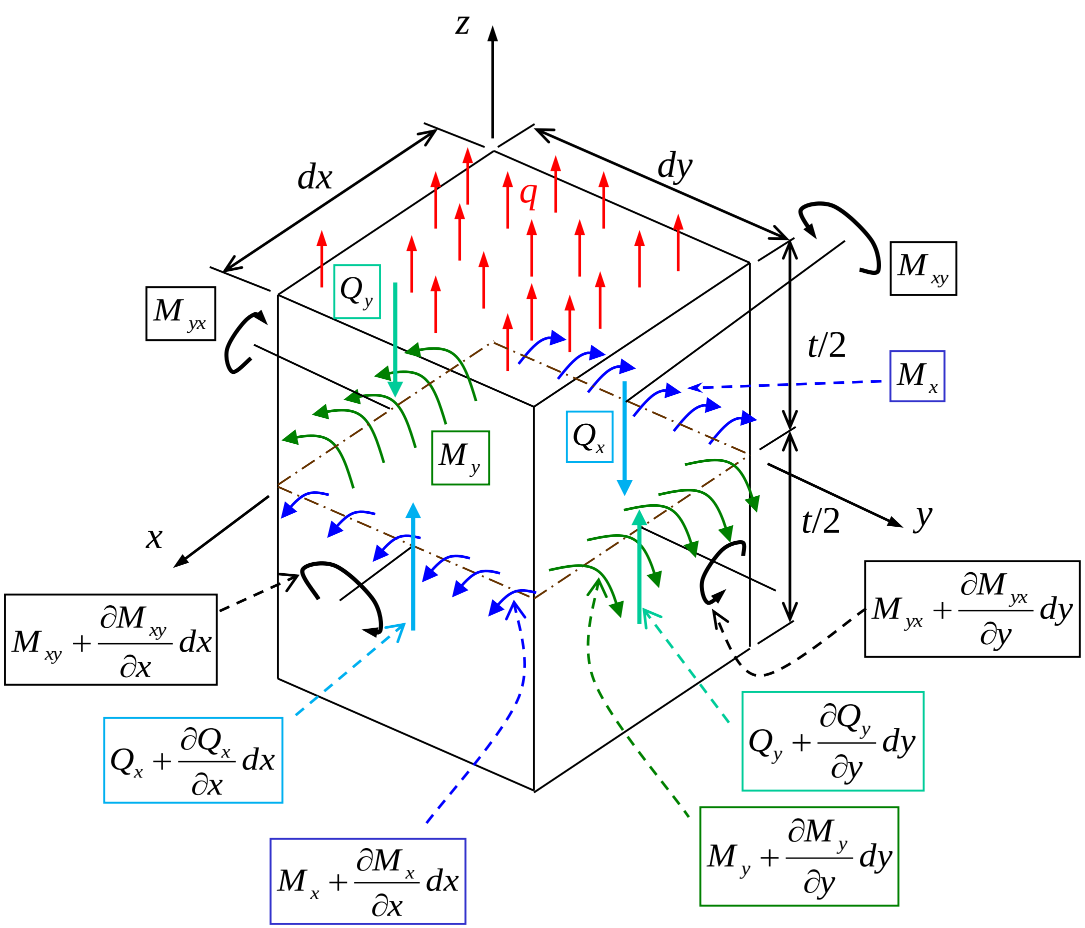


$$N_x = \int_{-t/2}^{t/2} \sigma_x dz$$

$$N_y = \int_{-t/2}^{t/2} \sigma_y dz$$

$$N_{xy} = N_{yx} = \int_{-t/2}^{t/2} \tau_{xy} dz$$

Plate infinitesimal element: internal moments and shear forces



$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_y z dz$$

$$M_{xy} = M_{yx} = \int_{-t/2}^{t/2} \tau_{xy} z dz$$

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz$$

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz$$

Force equilibrium equations

Force equilibrium along x

$$\left(N_x+\frac{\partial N_x}{\partial x}dx\right)dy-N_x dy+\left(N_{yx}+\frac{\partial N_{yx}}{\partial y}dy\right)dx-N_{yx}dx=0 \quad\Rightarrow\quad \frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0$$

Force equilibrium along y

$$\left(N_y+\frac{\partial N_y}{\partial y}dy\right)dx-N_y dx+\left(N_{xy}+\frac{\partial N_{xy}}{\partial x}dx\right)dy-N_{xy}dy=0 \quad\Rightarrow\quad \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0$$

Force equilibrium along z

$$\left(Q_x+\frac{\partial Q_x}{\partial x}dx\right)dy-Q_x dy+\left(Q_y+\frac{\partial Q_y}{\partial y}dy\right)dx-Q_y dx+q\,dxdy=0 \quad\Rightarrow\quad \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0$$

Moment equilibrium equations

Moment equilibrium about *x*

$$M_{xy}dy - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}dx \right)dy + M_y dx - \left(M_y + \frac{\partial M_y}{\partial y}dy \right)dx +$$

$$\left(Q_y + \frac{\partial Q_y}{\partial y}dy \right)dxdy + \left(Q_x + \frac{\partial Q_x}{\partial y}dx \right)\frac{(dy)^2}{2} - Q_x \frac{(dy)^2}{2} + qdx\frac{(dy)^2}{2} = 0$$

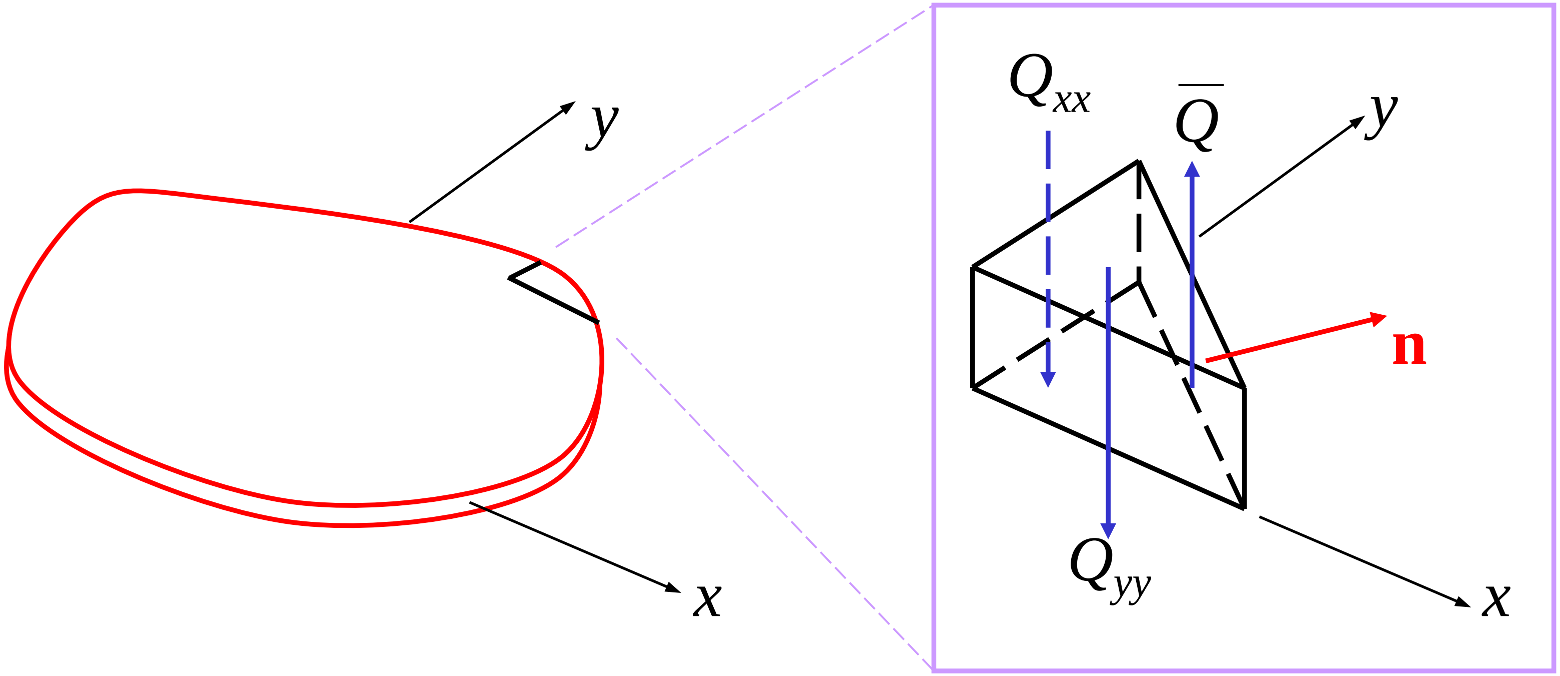
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$

Moment equilibrium about *y*

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0$$

Natural boundary conditions: shear forces

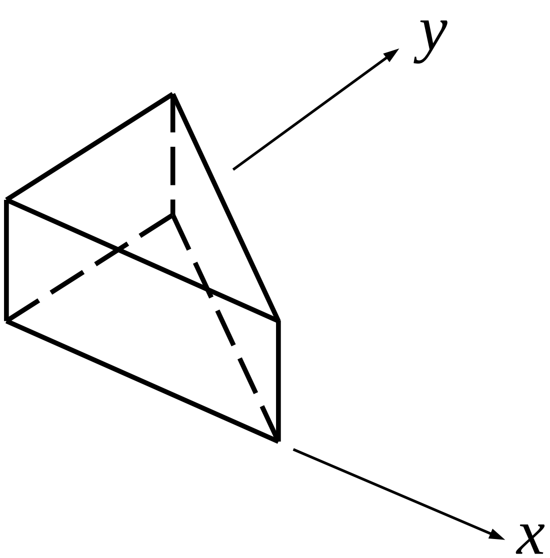
Prescribed boundary shear force: $\overline{Q}$

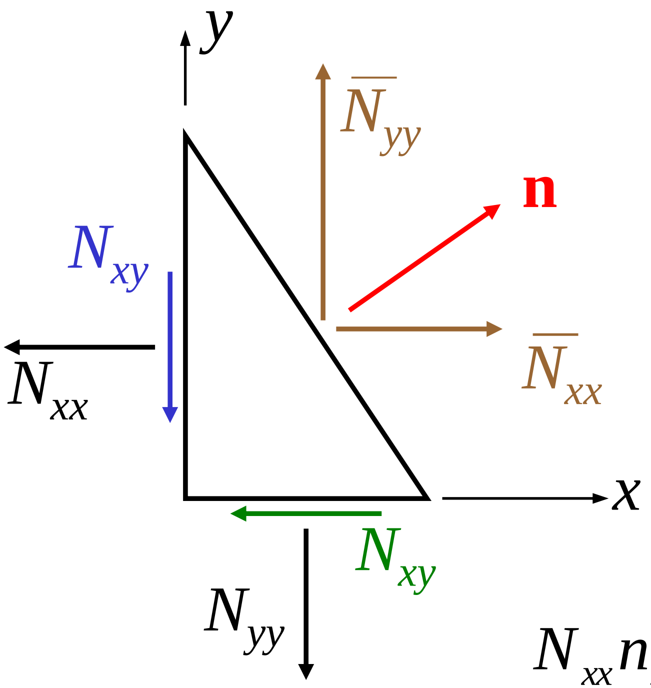


$$\overline{Q} = Q_{xx} n_x + Q_{yy} n_y$$

Natural boundary conditions: membrane forces

Prescribed boundary membrane forces: $\overline{N}_{xx}, \overline{N}_{yy}$



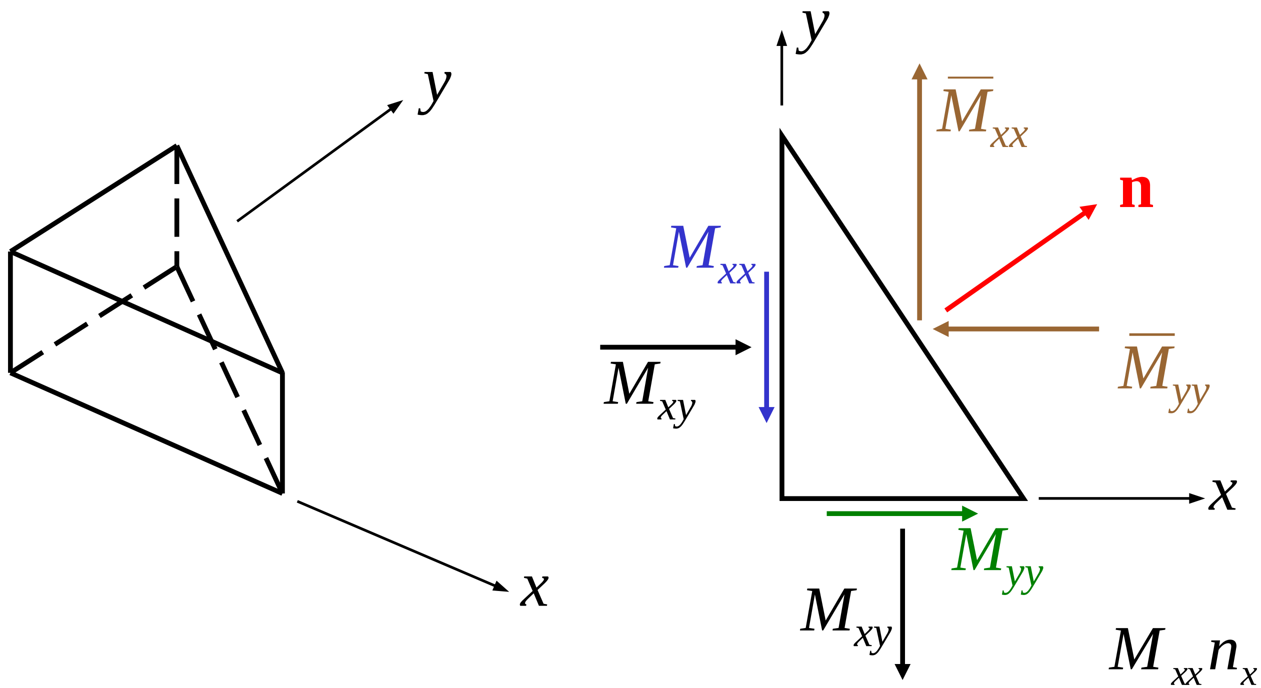


$$N_{xx}n_x + N_{xy}n_y = \overline{N}_{xx}$$

$$N_{xy}n_x + N_{yy}n_y = \overline{N}_{yy}$$

Natural boundary conditions: moments

Prescribed boundary moments: $\overline{M}_{xx}, \overline{M}_{yy}$



$$M_{xx}n_x + M_{xy}n_y = \overline{M}_{xx}$$

$$M_{xy}n_x + M_{yy}n_y = \overline{M}_{yy}$$

Strain × displacement relations

$$\varepsilon_{xx} = u_{,x} + z\psi_{x,x}$$

$$\varepsilon_{yy} = v_{,y} + z\psi_{y,y}$$

$$\gamma_{xy} = (u_{,y} + v_{,x}) + z(\psi_{x,y} + \psi_{y,x})$$

$$\varepsilon_{zz} = w_{,z} = 0$$

$$\gamma_{xz} = w_{,x} + \psi_x$$

$$\gamma_{yz} = w_{,y} + \psi_y$$

Constitutive relations for lamina k

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \qquad \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = [\bar{Q}_s]_k \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}_k$$

Equilibrium equations

$$N_{xx,x} + N_{xy,y} = 0$$

$$N_{xy,x} + N_{yy,y} = 0$$

$$Q_{xx,x} + Q_{yy,y} + q = 0$$

$$M_{xx,x} + M_{xy,y} - Q_{xx} = 0$$

$$M_{yy,y} + M_{xy,x} - Q_{yy} = 0$$

boundary conditions

Prescribed forces and moments

$$\overline{N}_{xx}, \overline{N}_{yy}, \overline{Q}, \overline{M}_{xx}, \overline{M}_{yy}$$

Prescribed displacements and rotations

$$U, V, W, \Theta_x, \Theta_y$$

Resultant laminate forces and moments

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} + z \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} \right) dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} + z \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} \right) zdz$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} w_{,y} + \psi_y \\ w_{,x} + \psi_x \end{Bmatrix} dz$$

Laminate matrices

$$[A]=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k dz = \sum_{k=1}^{N} (z_k - z_{k-1}) \left[\overline{Q}\right]_k$$

$$[B]=\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k z\,dz = \frac{1}{2} \sum_{k=1}^{N} (z_k^2 - z_{k-1}^2) \left[\overline{Q}\right]_k$$

$$[D]=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k z^2 dz = \frac{1}{3} \sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \left[\overline{Q}\right]_k$$

$$[A_s]=\begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_k dz = \sum_{k=1}^{N} (z_k - z_{k-1}) \left[\overline{Q}_s\right]_k$$

4.4. Special cases of laminates

Single layered configurations

Single isotropic layer

$$[A] = A\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \qquad [B]=[0] \qquad [D] = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$$A = \frac{Et}{1-\nu^2} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

Single specially orthotropic layer

$$[A] = t\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \qquad [B]=[0] \qquad [D] = \frac{t^3}{12}\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

Single orthotropic layer

$$[A] = t\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \qquad [B]=[0] \qquad [D] = \frac{t^3}{12}\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}$$

Symmetric laminates

Symmetric laminates are symmetric with respect to both geometry and material

For every layer *k* there must be another layer *k*' symmetrically located about the mid plane with the same material and fiber orientation angle. Notice that it is possible to have symmetric laminates with an odd number of layers

In this case it is easy to show that [*B*] = [0]. Therefore, there is no membrane-bending coupling

Unsymmetric laminates present strong curvatures after cure

Anti-symmetric laminates

Usually coupling between bending and extension must be avoided

Special applications where coupling is required

Quasi-isotropic laminates

Laminates that possess isotropic extensional stiffness

$$\left[0/\frac{\pi}{n}/\frac{2\pi}{n}/\ldots/\frac{(n-1)\pi}{n}\right]$$

Balanced laminates

Pairs of layers $\pm\theta$ with same thickness, not necessarily symmetric

Hybrid laminates

Mixture layers of two or more different materials

Matrices must be cure compatible

Cross-ply laminates

All layers at 0^o or 90^o

Angle-ply laminates

All layers at $-\alpha$ or $+\alpha$

4.5. Hygrothermal stresses

Hygrothermal effects

Purely mechanical analyses are insufficient to describe the behavior of laminates subject to temperature gradients

Thermal expansion coefficients must be known

$$\underbrace{\{\varepsilon\}}_{\text{total strain}} = \underbrace{[S]\{\sigma\}}_{\text{mechanical strain}} + \underbrace{\Delta T\{\alpha\}}_{\text{thermal strain}} + \underbrace{\Delta c\{\beta\}}_{\text{hygroscopic strain}}$$

$$\{\sigma\} = [\overline{Q}](\{\varepsilon\} - \Delta T\{\alpha\} - \Delta c\{\beta\})$$

Thermal effects

Orthotropic lamina in plane stress

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\left(\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}-\Delta T\begin{Bmatrix}\alpha_1\\ \alpha_2\\ 0\end{Bmatrix}\right)$$

Transformation into structural coordinate system

$$\begin{bmatrix}c^2 & s^2 & 2cs\\ s^2 & c^2 & -2cs\\ -cs & cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\left(\begin{bmatrix}c^2 & s^2 & cs\\ s^2 & c^2 & -cs\\ -2cs & 2cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}-\Delta T\begin{Bmatrix}\alpha_1\\ \alpha_2\\ 0\end{Bmatrix}\right)$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\left(\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}-\begin{bmatrix}c^2 & s^2 & cs\\ s^2 & c^2 & -cs\\ -2cs & 2cs & c^2-s^2\end{bmatrix}^{-1}\Delta T\begin{Bmatrix}\alpha_1\\ \alpha_2\\ 0\end{Bmatrix}\right)=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\left(\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}-\Delta T\begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{Bmatrix}\right)$$

$$\begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{Bmatrix}=\begin{bmatrix}c^2 & s^2 & -cs\\ s^2 & c^2 & cs\\ 2cs & -2cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ 0\end{Bmatrix}=\begin{Bmatrix}\alpha_1c^2+\alpha_2s^2\\ \alpha_1s^2+\alpha_2c^2\\ 2cs(\alpha_1-\alpha_2)\end{Bmatrix}$$

Thermal forces and moments

Integration through the thickness

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k dz$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k zdz$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix}$$

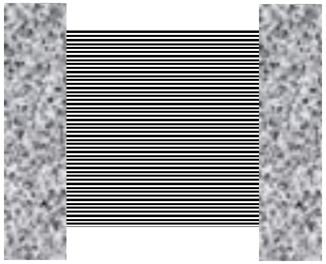
$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix}$$

Thermal effects

In virtually all laminates thermal effects cause residual thermal stresses because of the mismatch in thermal expansion coefficients from one lamina to the others

If the laminate is completely free there are no thermal residual membrane forces or moments, i.e., $[A]\{\varepsilon\} + [B]\{\kappa\} - \{N^T\} = \{0\}$ and $[B]\{\varepsilon\} + [D]\{\kappa\} - \{M^T\} = \{0\}$. This is usually the condition of the laminate right after curing

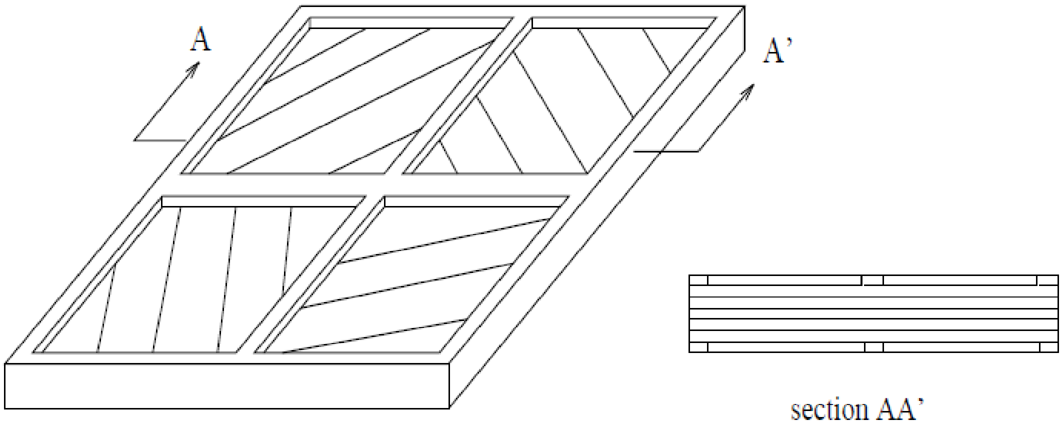
If the laminate is constrained thermal residual forces and moments might arise

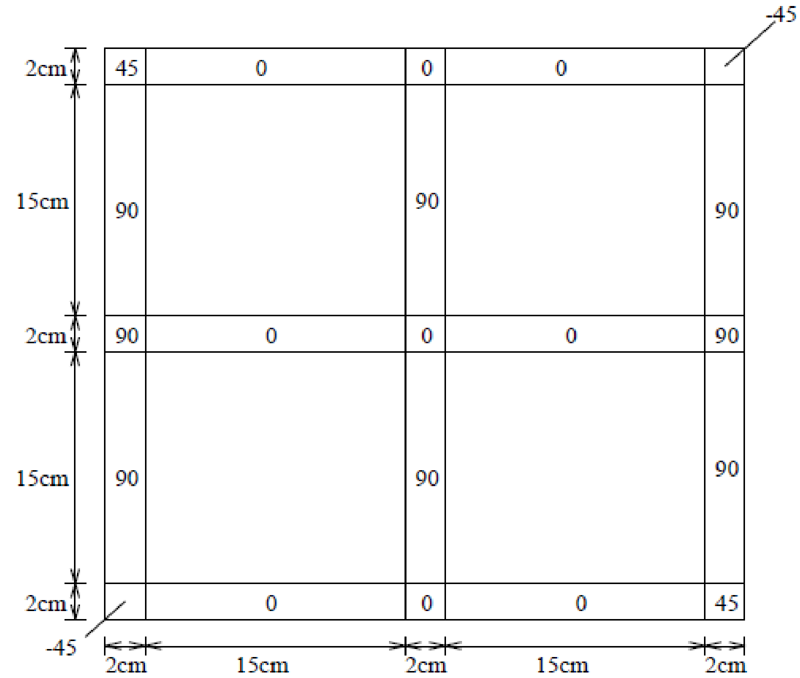


$\Delta T \neq 0$

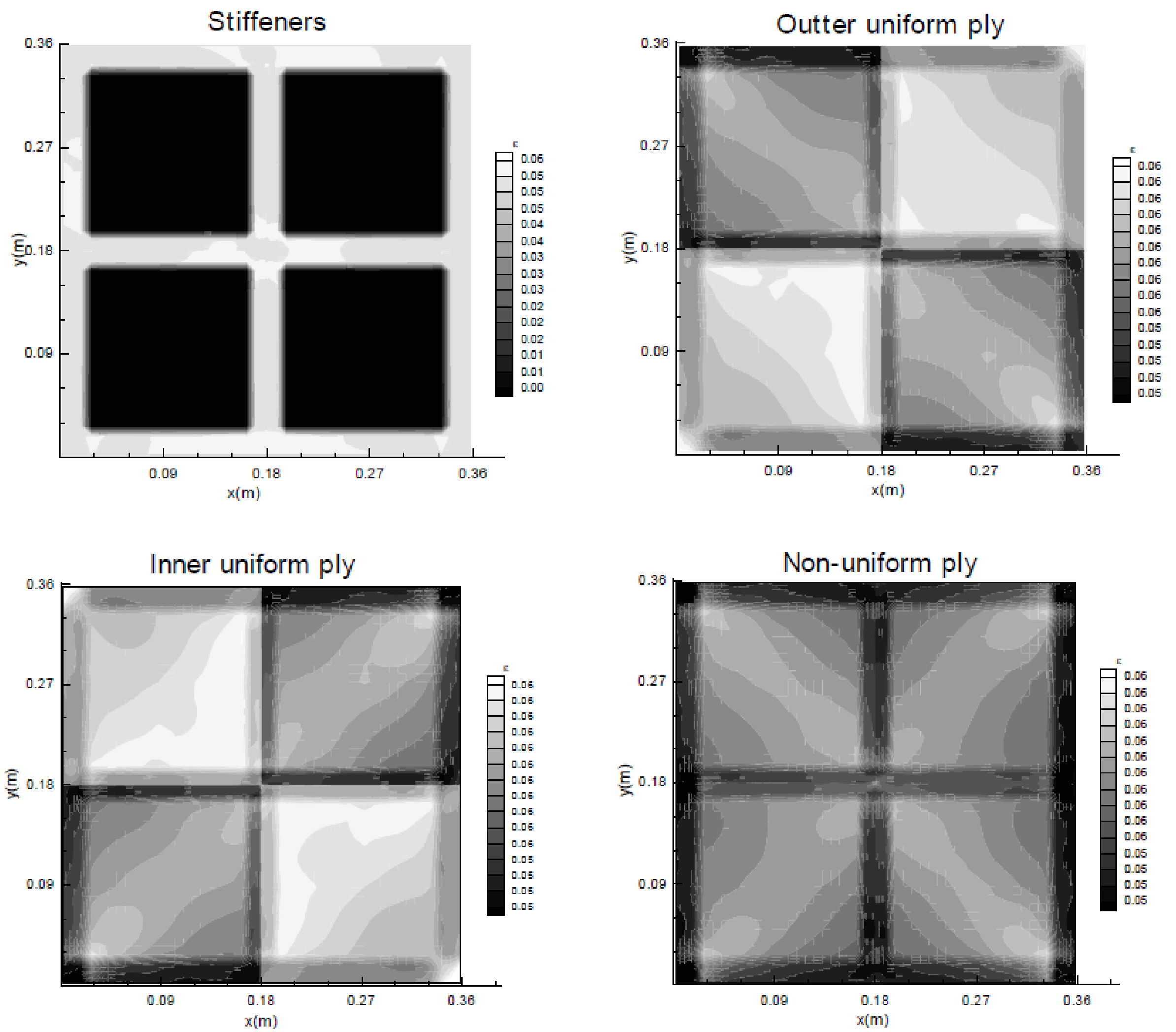
Thermal effects

Even in completely free heterogeneous laminates thermal residual stresses will arise





Thermal effects



Strength of a cross-ply laminate

E_1 = 53.78 GPa, E_2 = 17.93 GPa

ν_{12} = 0.25, G_{12} = 8.62 GPa

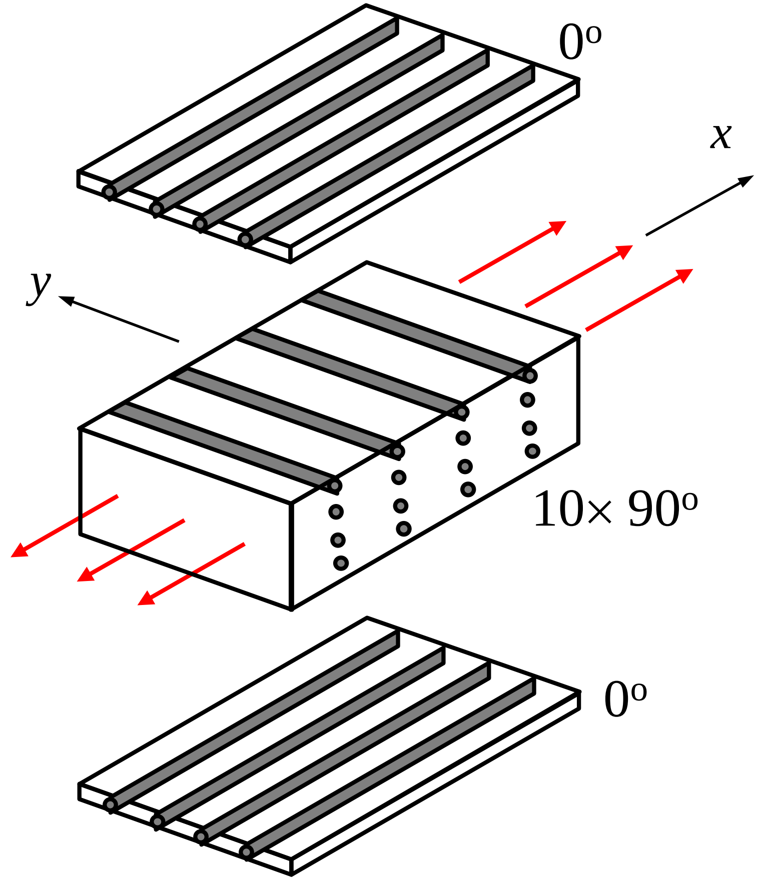
$\alpha_1 = 6.3\times10^{-6}$ °C^{-1}, $\alpha_2 = 20.52\times10^{-6}$ °C^{-1}

$X_t = X_c$ = 1035 MPa, Y_t = 27.6 MPa

Y_c = 138 MPa, S = 41.4 MPa

Two 0° layers and ten 90° layers

Layer thickness: 0.127 mm



Pre-failure deformation

$$[\bar{Q}]_0 = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \qquad [\bar{Q}]_{90} = \begin{bmatrix} Q_{22} & Q_{12} & 0 \\ Q_{12} & Q_{11} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

$$\alpha_x^0 = \alpha_y^{90} = \alpha_1$$
$$\alpha_y^0 = \alpha_x^{90} = \alpha_2$$
$$\alpha_{xy}^0 = \alpha_{xy}^{90} = 0$$

A_{11} = 0.037207 GN/m $\quad A_{22}$ = 0.074405 GN/m

A_{12} = 0.0069767 GN/m $\quad A_{66}$ = 0.013137 GN/m

$N_x^T = 0.41049\, t\, \Delta T$ MPa/ºC, $N_y^T = 0.43407\, t\, \Delta T$ MPa/ºC, $N_{xy}^T = 0$

$M_x^T = M_y^T = M_{xy}^T = 0$

$$\sigma_x^0 = 2.27 N_x / t + 0.4409 \Delta T \qquad \sigma_x^{90} = 0.75 N_x / t - 0.08819 \Delta T$$
$$\sigma_y^0 = 0.12 N_x / t - 0.1977 \Delta T \qquad \sigma_y^{90} = -0.024 N_x / t + 0.03954 \Delta T$$
$$\tau_{xy}^0 = 0 \qquad \tau_{xy}^{90} = 0$$

Tsai-Hill failure criterion

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1\sigma_2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\tau_{12}^2}{S^2} = 1 \quad \Longrightarrow \quad \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 (X/Y)^2 = X^2$$

0° layer: $\frac{N_x}{t} = 1.365\Delta T + \sqrt{57.5Y^2 - 0.4621(\Delta T)^2}$ $\quad Y$ [MPa] and ΔT [°C]

A) Cure at 132°C and laminate used at 21°C $\Rightarrow \Delta T = -111$°C

0° layer: N_x/t = 43.37 MPa

90° layer: N_x/t = 23.44 MPa

B) Cure at 21°C and laminate used at 21°C $\Rightarrow \Delta T = 0$°C

0° layer: N_x/t = 209.3 MPa

90° layer: N_x/t = 36.68 MPa

ε_x = 0.098%